In presenting the dissertation as a partial fulfillment of the requirements for an advanced degree from the Georgia Institute of Technology, I agree that the Library of the Institute shall make it available for inspection and circulation in accordance with its regulations governing materials of this type. I agree that permission to copy from, or to publish from, this dissertation may be granted by the professor under whose direction it was written, or, in his absence, by the Dean of the Graduate Division when such copying or publication is solely for scholarly purposes and does not involve potential financial gain. It is understood that any copying from, or publication of, this dissertation which involves potential financial gain will not be allowed without written permission.

7/25/68

# TRAJECTORY OPTIMIZATION FOR THE COMBINED ESTIMATION AND CONTROL OF NONLINEAR STOCHASTIC SYSTEMS

A THESIS

Presented to

The Faculty of the Graduate Division

by

Robert Jordan Brown, Jr.

In Partial Fulfillment

of the Requirements for the Degree

Doctor of Philosophy

in the School of Electrical Engineering

Georgia Institute of Technology

May, 1971

TRAJECTORY OPTIMIZATION FOR THE COMBINED ESTIMATION AND CONTROL OF NONLINEAR STOCHASTIC SYSTEMS

Approved:

Chairman

Date approved by Chairman: May 11, 1971

## ACKNOWLEDGEMENTS

To Dr. James R. Rowland, my thesis advisor, I would like to express much appreciation for his guidance and encouragement throughout my graduate study at Georgia Tech and especially during the development of this dissertation. I have enjoyed many hours of stimulating discussion with Dr. Rowland in the course of planning and executing this research. Appreciation is also extended to Drs. Aubrey M. Bush and Jay H. Schlag for their services as members of the reading committee.

I would like to extend special thanks to the Schlumberger Foundation, Lockheed-Georgia Company, and the National Science Foundation for providing me with fellowships over the past three years.

To my parents, who have always encouraged me in my endeavors, go my thanks and sincere appreciation.

Finally, to my wife Martha goes my love and deepest appreciation for her patience, understanding, and love during my graduate study at Georgia Tech. I would also like to express my thanks to her for her help in typing and preparing this dissertation manuscript.

TABLE OF CONTENTS

# LIST OF ILLUSTRATIONS

## SUMMARY

This dissertation presents an improved approximate solution for the nonlinear combined estimation and control problem. The system equations may be nonlinear in the state, control, and measurement but are assumed to be expandable in a Taylor series. The plant and measurement noises are assumed to be independent white noise processes which enter linearly into the system equations but may be nongaussian in general. A quadratic performance index of the Bolza form is chosen as the criterion for determining system performance.

The approach of trajectory optimization is used in developing an improved suboptimal solution to the combined estimation and control problem for nonlinear systems. The method involves optimizing simultaneously the system nominal control and nominal trajectory along with other parameters of the control system. Following a treatment of the open-loop problem, the closed-loop problem with no measurement noise is considered. Finally, an improved suboptimal solution for the nonlinear combined estimation and control problem is presented for both linear and nonlinear filtering in the control loop. In developing the solution for the three cases, the stochastic optimization problem is transformed into a deterministic optimization problem in the

calculus of variations. Although the approach is approximate, second-order stochastic effects are included in all equations. Another important feature of the method is that a separation principle is not arbitrarily invoked in the treatment of the estimation and control problem.

It is demonstrated through specific nonlinear examples for all three cases studied that significant improvement over the widely used method of simply linearizing the system equations about the deterministic optimal trajectory can be realized. A major conclusion for the combined estimation and control problem is that the system performance is apparently much more sensitive to the choice of the nominal trajectory than to the selection of a nonlinear filter producing greater estimation accuracy.

# CHAPTER I

# INTRODUCTION

## Introduction and Problem Statement

The optimal control of nonlinear stochastic systems is a major problem in systems theory and as yet has not been solved in a suitable framework to allow many practical applications of results. However, most practical systems are inherently nonlinear and subject to stochastic disturbances. Furthermore, the system state is often not exactly measurable because of the presence of noise in the measurement subsystem. A suitable filter must then be designed to produce an estimate of the state. It would appear, therefore, that an improved practical solution for nonlinear stochastic systems is often needed. When both plant and measurement noise are present in a control system, the problem is generally referred to as the combined estimation and control problem. This thesis describes the development of an improved approximate solution to this combined optimization problem for nonlinear systems.

The general configuration of the system considered in this research is shown in Figure 1. For the given system, the plant equation is

$$\dot{x} = f(x,u,t) + w \tag{1.1}$$

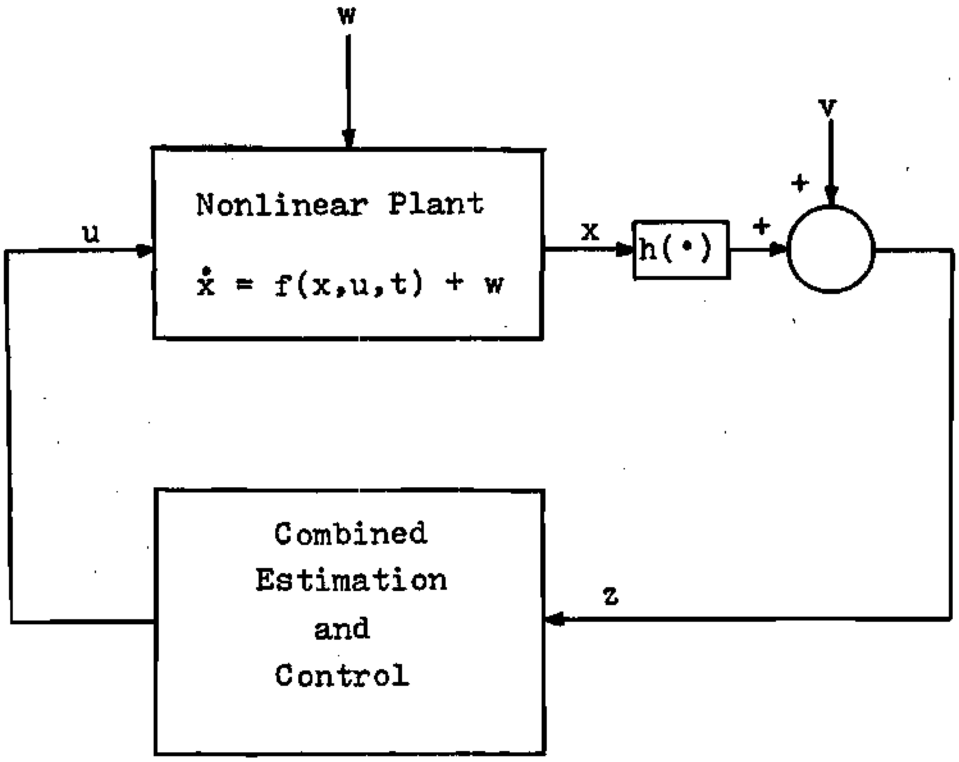


Figure 1. The Nonlinear Estimation and Control Problem

where x represents the n-dimensional state vector, u is the m-dimensional control vector, and w is the n-dimensional vector representing white noise with zero mean and covariance given by

$$E[w(t)w^T(\tau)] = Q_w\delta_d(t-\tau) \tag{1.2}$$

where $E[\cdot]$ denotes the expected value operation, and $\delta_d(\cdot)$ is the Dirac delta function. The noise w is assumed to be nongaussian in general and to be entering linearly into the plant equation. The observation vector of the system is given by

$$z = h(x) + v \tag{1.3}$$

where z is the r-dimensional vector of measurements, h(x) is a nonlinear function of the state x, and v is the r-dimensional vector of additive white noise with zero mean and covariance

$$E[v(t)v^T(\tau)] = Q_v\delta_d(t-\tau) \tag{1.4}$$

The performance of the system is measured by a quadratic index of the Bolza form given by

$$J = E[\ \tfrac{1}{2}x^T(t_f)Sx(t_f) + \int_{t_o}^{t_f} (\tfrac{1}{2}x^TQ_1x + \tfrac{1}{2}u^TR_1u)dt] \tag{1.5}$$

The terminal time $t_f$ is assumed to be fixed. The matrices S, $Q_1$, and $R_1$ are positive semidefinite, symmetric matrices which may, in general, be time-varying. The initial state is assumed known, i.e. $x(t_o) = x_o$. The design objective in the combined estimation and control problem is to minimize J by selecting the appropriate estimator and controller.

In general, the optimization of nonlinear stochastic systems simply cannot be performed exactly, and suitable approximations to the solution must be utilized. The main emphasis of this thesis research has been the development of a trajectory optimization technique which yields an improved approximate solution to the nonlinear combined estimation and control problem.

## Exact Solutions for Stochastic Systems

Three aspects of general stochastic systems theory are described in this section. The filtering problem, which has become a major area of investigation in recent years, is considered first. The stochastic control problem with exact knowledge of the state is discussed. The filtering and control problems are then tied together through an examination of previous results for the combined estimation and control problem.

### The Filtering Problem

One encounters the stochastic filtering problem when the observation of some physical process is corrupted by random disturbances. The problem is to produce the best estimate of the state of a system based on some criterion which reflects the goodness of the estimate. In the early 1940's Wiener [1] developed some basic linear results in stochastic theory in his search for a rational design for fire-control systems. Wiener assumed that the state and measurement were jointly wide-sense stationary, ergodic processes with the observation of the system being given from $t_o = -\infty$. His theory resulted in a minimum mean-square error estimate of the state x(t). The estimator was linear, and the weighting matrix, or state transition matrix, satisfied an integral equation of the Wiener-Hopf type [1]. The analagous discrete-time filtering problem was subsequently solved by Kolmogorov [2].

Various investigators derived generalized scalar Wiener-Hopf equations for nonstationary processes with finite observation time. However, the practical solution of these equations has not yet been accomplished. The Kalman-Bucy theory [3], which could handle linear nonstationary processes, appeared in 1961. Instead of solving the Wiener-Hopf equation, a stochastic differential equation was derived for the estimate. The central idea was to solve a matrix Riccati equation which contained the error covariance of the estimate. The estimate in the Kalman-Bucy theory was optimal if the system were linear and all noise disturbances gaussian. The Kalman-Bucy theory has since found widespread application in such areas as guidance, navigation, and orbit determination. Some of these applications were applied to Mariner, Ranger, Apollo, and LEM back-up guidance.

In many practical situations, the noise disturbances are not gaussian, and the state equations are not linear. In either case, the optimal estimator is not linear. The difficulty encountered in nonlinear filtering problems is the determination of the evolution of the conditional probability density function of the plant state given the measurements. The nonlinear filtering problem was solved theoretically by Stratonovich [4] in 1960. He obtained a random partial differential equation for the conditional density which was a modified Fokker-Planck equation. Kushner [5,6] independently derived the same result, and Bucy [7] later rederived Kushner's result using stochastic calculus. This same approach was also taken by Wonham [8], Kashyap [9], and Fisher [10]. The probabilistic approach to discrete-time filtering was developed by Ho and Lee [11]. Jazwinski [12] treated the continuous-discrete problem. It should be noted here that although these authors derived exact equations for the evolution of the conditional density function of the state, only the equations for linear systems with gaussian disturbances have been solved. For this reason, approximate solutions are employed for the nonlinear filtering problem.

The Stochastic Control Problem With No Measurement Noise

The objective in the stochastic control problem, in the case where the state vector is exactly known, is to minimize a performance index such as (1.5) by selecting a control u as a function of the state x. The resulting solution will be a closed-loop feedback configuration. Only a few results are available for stochastic systems even when there is perfect knowledge of the plant state. The formulation of stochastic optimal control problems is due largely to Bellman [13]. Although the Maximum Principle [14] is available for the treatment of deterministic control problems, there does not exist at present an adequate theory for handling stochastic problems. Some work in this direction has been performed by Mischenko and Pontryagin [15] and Kolmogorov et al [16]. Florentin [17] derived a nonlinear integro-partial differential equation whose solution yields the optimal control as well as the value of the performance index. However, only when the plant is linear and the cost criterion quadratic can the resulting equation be solved. In the analysis of the general nonlinear stochastic control problem, Wonham [18] obtained a Kolmogorov partial differential equation by applying Hamilton-Jacobi theory to the expected value of a given performance criterion. Although the Kolmogorov equation is very difficult, if not impossible to solve in general, the problem for linear systems with quadratic cost may be reduced to the solution of an ordinary differential equation of the Riccati type [18]. In this case, the feedback control is precisely the same control as for the system when no noise is present. General conditions for which the stochastic control and the deterministic control are identical have not been determined. Very little is known about conditions under which a solution to the stochastic control problem exists. Wonham [18] states that exact solutions in stochastic control theory are still quite far away from the stage of practical application. In practical cases, approximate solutions are almost always utilized.

Combined Estimation and Control

Additional complications arise if the state variables are not known exactly and must be estimated. When measurement noise is present, the difficulty lies in the infinite dimensional nature of the combined estimation and control problem [19] which arises because of the need to know the equivalent of all higher order moments of the plant state. Control action in the general nonlinear stochastic problem serves not only the purpose of transfering the plant to a desired state, but also aids in gaining information about the plant state. In recognition of the dual purpose of the control, Feldbaum [20] referred to the combined optimization problem as the dual control problem.

One very central question that arises in the combined estimation and control problem is whether a separation principle holds. The existence of a separation principle allows one to obtain the optimal overall system by cascading the optimal filter and the optimal deterministic controller. Joseph and Tou [21] and Alspach and Sorenson [22] have shown that when the performance criterion is quadratic and the plant linear, the control and estimation problems may be considered separately. The optimal controller is the Riccati controller obtained by neglecting all noises. If the noises are gaussian, the optimal estimator is the Kalman filter. Curry [23] developed a separation principle for discrete linear systems with nonlinear measurements. For more general nonlinear discrete time systems with arbitrary cost criteria, Striebel [24] was able to show that the conditional probability distribution of the system state given the measurement was sufficient to derive the optimal control. If the filtering problem were viewed as the calculation of the conditional probability distribution, one would have then a form of separation. Wonham [25] has shown that a form of separation is valid for continuous linear systems with arbitrary fixed terminal time and with controls that satisfy a Lipshitz condition. Gran [26] extended Wonham's result to discrete-time linear systems with bang-bang controls and free terminal time. Wonham and Gran showed that separation in the linear system with arbitrary performance criteria is one-sided. The optimal filter is indeed the Kalman filter with a control term added, but the optimal solution is not given by the noise-free controller, but rather by a stochastic controller whose character depends upon the Kalman filter. Gran [27] extended Curry's result for nonlinear measurements to the continuous case. For general nonlinear plants, very little is known about the structure of the optimal solution. The Separation Principle is often arbitrarily invoked and approximations employed to obtain useful suboptimal solutions.

## Approximate Techniques for Nonlinear Stochastic Systems

It was noted that exact solutions are not generally available for nonlinear stochastic systems. Meier [19] stated that investigation of approximations is a prime area of unfinished research on even the classical optimal control problem and especially for the combined optimization problem. Recent approximate results for stochastic systems are summarized in this section.

### Approximate Nonlinear Filtering

The solution of the nonlinear filtering problem could be implemented exactly if the Fokker-Planck equation could be solved. It is often sufficient in minimum variance estimation to calculate only the conditional mean and covariance of the state. Equations for these moments are obtainable from the Fokker-Planck equation, but are stochastic differential equations which have not been solved except for the linear gaussian case.

The difficulties encountered in the exact solution of the nonlinear filtering problem force one to consider approximations as a suboptimal solution to the problem. A parametrization of the density function in an orthogonal series expansion has been suggested as an approximate solution to the problem. The Edgeworth series expansion has been used by Sorenson and Stubberud [28] in developing approximate filters for the scalar, discrete system. A related method of parametrizing the conditional density is expanding the density function via moments. Kuo [29,30] employed a moment technique to approximately solve the nonlinear filtering problem for linear systems with nongaussian noise.

Probably the most common approximate method for nonlinear filtering is to expand the system message model in a series which is truncated after the first few terms. The series may then be substituted into the equations for the conditional mean and covariance of the state derived from the Fokker-Planck equation. This technique was described by Jazwinski [31] and by Sage and Melsa [32]. Depending upon how many terms are retained, either a first, second, or higher order approximate filter is obtained. A linear filter is obtained if the nonlinear system is expanded about a deterministic nominal trajectory, and a perturbation estimator developed for the linear perturbation equation [31,32]. Probably the most widely used nonlinear filter is the extended Kalman filter obtained when the message model is expanded about the current estimate of the state [31,32].

A truncated second-order filter was developed by Jazwinski [33,34] and independently by Bass et al [35]. Schwartz [36], Jazwinski [34], and Fisher [10] independently developed the gaussian second-order filter. The term gaussian is applied to the filter because the density function is assumed to be approximately gaussian. Sunahara [37] proposed replacing the nonlinear functions in the state and measurement equations by quasilinear functions via stochastic linearization. However, this technique has not been tested computationally in nonlinear filtering problems [31]. Schwartz and Stear [38] reported simulations of several nonlinear filters (gaussian second-order, extended Kalman, and others), but none were consistently superior in performance. However, the extended Kalman filter was recommended on the basis of its simplicity and small computational requirements. Jazwinski [31] noted that the performance of the extended Kalman filter may be improved by iterating the filter over the estimates and data repeatedly. However, this operation is usually not feasible when the filter is applied to a real-time control situation.

Approximate Solutions for Estimation and Control

Rather than solve the Kolmogorov equation for the nonlinear control problem with no measurement noise, a useful approximate technique is to expand the nonlinear system equations about the deterministic optimal trajectory. If only linear terms are included in the expansion, a Riccati controller may be employed about the nominal. This technique has been described by Bryson and Ho [39]. Kushner [40] described a technique whereby increased accuracy may be obtained by expanding the series using higher-order terms. Culver and Mesarovic [41] used dynamic statistical linearization theory to approximate a nonlinear system by a general dynamic linear operator defined over the class of system inputs.

If the added complexity of measurement noise is introduced into the system, one has the combined estimation and control problem. Wonham [18] noted that no satisfactory theory of optimal control exists for the case including measurement noise. To handle the problem on an approximate basis, the nonlinear system may be linearized as before about the deterministic optimal trajectory. A Kalman filter may be employed to estimate perturbations about the nominal, and a Riccati perturbation controller used to add corrective terms to the nominal control. Sunahara [42] extended his method of stochastic linearization to obtain an approximate solution to the combined optimization problem for nonlinear systems with state-dependent noise. Because the method was not applied to a particular system, the accuracy of Sunahara's result is not known. By far the most widespread approximate method for nonlinear systems is to apply well-known linear techniques to a linearized version of the problem.

## Trajectory Optimization

It has been noted that when a nonlinear system is expanded about a nominal trajectory, that nominal is usually taken as the deterministic optimal trajectory. However, Kushner [40] was perhaps the first to observe that the deterministic optimal control schedule is often not the best nominal when noise is present. Bryson and Ho [39] and Denham [43] also indicated that the best nominal is not necessarily the deterministic optimal trajectory. In considering the stochastic control problem with no measurement noise, Kushner [40] developed a perturbation feedback control technique which also produced a better nominal trajectory than the deterministic optimal trajectory. However, his method was practical only if noise disturbances occurred at only two points in time during the control interval. Severe complications arose in the method's application to continuous noise disturbances. Denham observed that the statistical effectiveness of the feedback control system depends upon the nominal path and that often better performance may be obtained when the nominal trajectory and feedback control system are simultaneously chosen to optimize a statistical measure of the overall system performance. Vander Stoep [44] considered the problem of choosing the nominal trajectory to minimize estimation errors propagating along the nominal trajectory. His approach was to augment the original performance index by including terms which are functions of the estimation error covariance. Although his method failed to yield the nominal which minimized the original performance index, it did offer interesting tradeoffs between system performance and the performance of the perturbation estimator. Meier, et al [45] treated the problem of finding the best nominal by first approximating the expected value of a Bolza-type performance index and then assuming an approximate solution to the Hamilton-Jacobi equation for the return function. On the other hand, Denham [43] treated the problem of minimizing the expected value of a function of the state evaluated at the final time. Denham derived a two-point boundary value problem which would approximately yield the best nominal control, feedback gain matrix, and linear filter parameters. Fitzgerald [46] also considered the problem of shaping the nominal trajectory. Using a somewhat less general performance index than Denham had used, Fitzgerald treated more complex noise models which were state-dependent. Fitzgerald failed to consider the cases requiring feedback control and state estimation. One of the main points of his work was the clever modification of the gradient method to solve the optimization problem.

Denham, Fitzgerald, Vander Stoep, and Meier recognized the need for simultaneously optimizing the nominal trajectory along with other control and filter parameters when designing nonlinear stochastic systems. Each made simplifying approximations at various points in their research.

These differing approximations as well as the lack of subsequent nonlinear examples have made meaningful comparisons difficult. This research investigated the nonlinear combined estimation and control problem via a somewhat different approach. Systematic approximations were utilized in the present development, and specific nonlinear examples were worked to show the improvement in performance realized over previous approximate methods by the new trajectory optimization method.

## Outline of the Thesis

The major emphasis in this research was the optimization of the system nominal trajectory to minimize a given performance index. The approach to the optimization of the nominal trajectory was to convert the stochastic optimization problem to a deterministic optimization problem in the calculus of variations. Although the approach was approximate, a significant improvement in system performance was realized over the performance obtained by simply linearizing about the deterministic optimal trajectory. The most important conclusion of this research is the observation that the performance of the overall estimation and control system is apparently much more sensitive to the choice of a better nominal trajectory than to the selection of a more accurate state estimator.

In Chapter II, the new approach to trajectory optimization is introduced for the open-loop stochastic system. In the open-loop case, it is assumed that only an open-loop control schedule is required. The trajectory optimization approach is used to determine the approximately optimal open-loop control for the nonlinear stochastic system. In Chapter III, the application of trajectory optimization to the closed-loop system with no measurement noise is described. In the closed-loop case, a nominal trajectory, nominal control, and perturbation feedback controller are optimized simultaneously for the given quadratic performance index. The method of trajectory optimization is applied to the combined estimation and control problem in Chapter IV. For the closed-loop system with observation noise, a linear perturbation estimator is chosen along with the nominal trajectory, nominal control, and perturbation controller. In Chapter V, a nonlinear filter is used in the feedback loop to show what improvement may be realized in the overall performance. Finally, in Chapter VI conclusions are summarized, and recommendations for further work are made.

# CHAPTER II

# THE OPEN-LOOP PROBLEM

## Introduction

This chapter introduces the new approach to trajectory optimization by describing the results obtained for the open-loop problem. In the open-loop problem, it is assumed that a closed-loop system is not required and that only an open-loop control schedule is desired. The value of the trajectory optimization approach is illustrated by a nonlinear numerical example.

## Mathematical Development

The objective in the open-loop problem is to minimize the performance index (1.5) subject to the state constraint (1.1) and other constraints that are formulated in the course of the development of trajectory optimization. The immediate goal is to transform the stochastic optimization problem to a deterministic problem in the calculus of variations.

In the trajectory optimization approach, it is assumed that the plant state may be represented by

$$x = \bar{x} + \delta x \qquad (2.1)$$

where $\bar{x}$ is the deterministic nominal trajectory to be found, and $\delta x$ represents the stochastic deviation from the nominal trajectory caused by the plant noise w. Substituting (1.1) into (1.5) yields

$$J = E\left\{\tfrac{1}{2}[\bar{x}(t_f) + \delta x(t_f)]^T S[\bar{x}(t_f) + \delta x(t_f)] \right. \tag{2.2}$$
$$\left. + \int_{t_o}^{t_f} \tfrac{1}{2}(\bar{x} + \delta x)^T Q_1(\bar{x} + \delta x) + \tfrac{1}{2}u^T R_1 u \, dt\right\}$$

Taking the expected value operation inside the integral and noting that $\bar{x}$ and u are assumed deterministic, one has

$$J = \tfrac{1}{2}\bar{x}^T(t_f)S\bar{x}(t_f) + \bar{x}^T(t_f)SE[\delta x(t_f)] \tag{2.3}$$
$$+ \tfrac{1}{2} \text{ trace } SE[\delta x(t_f)\delta x^T(t_f)]$$
$$+ \int_{t_o}^{t_f} (\tfrac{1}{2}\bar{x}^T Q_1\bar{x} + \bar{x}^T Q_1 E(\delta x) + \tfrac{1}{2} \text{ trace } Q_1 E(\delta x \delta x^T) + \tfrac{1}{2}u^T R_1 u)dt$$

The substitutions

$$M = E(\delta x)$$
$$P = E(\delta x \delta x^T)$$

may be introduced into (2.3) to give

$$J = \tfrac{1}{2}\bar{x}^T(t_f)S\bar{x}(t_f) + \bar{x}^T(t_f)SM(t_f) + \tfrac{1}{2} \text{ trace } SP(t_f) \tag{2.4}$$
$$+ \int_{t_o}^{t_f} (\tfrac{1}{2}\bar{x}^T Q_1\bar{x} + \bar{x}^T Q_1 M + \tfrac{1}{2} \text{ trace } Q_1 P + \tfrac{1}{2}u^T R_1 u)dt$$

where the trace of a matrix is simply the sum of its diagonal elements. The only variables in this performance index are $\bar{x}$, M, and P. When differential equations for these variables have been obtained, the problem will reduce to a deterministic one in the calculus of variations.

The state equation for the plant may be expanded about the nominal $\bar{x}$ in a Taylor series as

$$\dot{x} = \dot{\bar{x}} + \delta\dot{x} = f(\bar{x},u,t) + \left(\frac{\partial f}{\partial x}\right)^T \Bigg|_{x=\bar{x}} + w \tag{2.5}$$

$$+ \text{ higher-order terms in } \delta x$$

If $\delta x$ is sufficiently small, the higher-order terms may be neglected so that (2.5) may be approximated by the two equations

$$\dot{\bar{x}} = f(\bar{x},u,t) \tag{2.6}$$

$$\delta\dot{x} = F\delta x + w \tag{2.7}$$

where

$$F = \left(\frac{\partial f}{\partial x}\right)^T \Bigg|_{x=\bar{x}}$$

The $\delta\dot{x}$ equation in (2.7) is linear, so that P approximately satisfies the equation

$$\dot{P} = FP + PF^T + Q_w \tag{2.9}$$

Note that (2.9) is only approximate because (2.7) was obtained by approximation. Equation (2.9) describes the evolution of the mean square values, including the cross-product terms, of all components of the vector $\delta x$, which is a well-known result because of the linearity of the perturbation equation (2.7). The matrix P is symmetric and, therefore determines $(n^2 + n)/2$ constraints.

The $\delta \dot{x}$ differential equation may be expanded to second-order. For second-order terms, matrix notation is cumbersome and, therefore, the equation for $\delta \dot{x}$ may be expressed in component form as

$$\delta \dot{x}_j = \sum_{i=1}^{n} \left. \frac{\partial f_j}{\partial x_i} \right|_{x=\bar{x}} \delta x_i + \frac{1}{2} \sum_{k=1}^{n} \sum_{i=1}^{n} \left. \frac{\partial^2 f_j}{\partial x_i \partial x_k} \right|_{x=\bar{x}} \delta x_i \delta x_k + w \qquad (2.10)$$

It is assumed that the expected value of $\delta \dot{x}_j$ equals the derivative of the expected value of $\delta x_j$. This assumption is valid if $f(x,u,t)$ has continuous partials through second-order, and if the white noise term w is not state-dependent [32]. Taking the expected value of both sides of (2.10), gives the equation for the evolution of the mean of $\delta x$ as

$$\dot{M}_j = \sum_{i=1}^{n} \left. \frac{\partial f_j}{\partial x_i} \right|_{x=\bar{x}} M_i + \frac{1}{2} \sum_{k=1}^{n} \sum_{i=1}^{n} \left. \frac{\partial^2 f_j}{\partial x_i \partial x_k} \right|_{x=\bar{x}} P_{ik} \qquad (2.11)$$

The open-loop nonlinear stochastic control problem has been reduced to the minimization of the deterministic performance index

$$J = \frac{1}{2}\bar{x}^T(t_f)S\bar{x}(t_f) + \bar{x}^T(t_f)SM(t_f) + \frac{1}{2}\text{ trace } SP(t_f) \tag{2.12}$$

$$+ \int_{t_o}^{t_f} (\frac{1}{2}\bar{x}^T Q_1\bar{x} + \bar{x}^T Q_1 M + \frac{1}{2}\text{ trace } Q_1 P + \frac{1}{2}u^T R_1 u)dt$$

subject to the constraints

$$\dot{\bar{x}} = f(\bar{x},u,t) \tag{2.13}$$

$$\dot{M}_j = \sum_{i=1}^{n} F_{ji}M_i + \frac{1}{2}\sum_{k=1}^{n}\sum_{i=1}^{n} F_{jik}P_{ik}\ ,\ j = 1,\ldots,n \tag{2.14}$$

$$\dot{P} = FP + PF^T + Q_w \tag{2.15}$$

where

$$F_{ji} = \left.\frac{\partial f_j}{\partial x_i}\right|_{x=\bar{x}} \qquad F_{jik} = \left.\frac{\partial^2 f_j}{\partial x_i \partial x_k}\right|_{x=\bar{x}}$$

The problem posed by (2.12)-(2.15) may be solved by adjoining the constraints (2.13)-(2.15) to the performance index (2.12) through Lagrange multipliers and forming the Hamiltonian. By performing the appropriate operations to obtain the canonical equations, a nonlinear two-point boundary value problem is obtained. It is assumed that the initial conditions are

$$\bar{x}(t_o) = x_o, \quad M(t_o) = 0, \quad \text{and} \quad P(t_o) = 0 \qquad (2.16)$$

The equations (2.16) imply that initially the system is unperturbed. Using (2.16) with the final conditions on the adjoint variables determined through the calculus of variations problem, $n^2 + 5n$ boundary conditions are obtained for the $(n^2 + 5n)/2$ differential equations (2.13)-(2.15) and the $(n^2 + 5n)/2$ adjoint differential equations. These equations may be solved by various conventional methods presently available. Although the equations are approximate, it was shown that system performance was improved by using the new open-loop control schedule u obtained from the solution of (2.12)-(2.16).

## Application to the General First-order System

The theory developed in the previous section is now illustrated for the general first-order system. A first-order plant is employed because notation in the formulation of the Hamiltonian is cumbersome for higher-order plants, and conclusions drawn from the treatment of the first-order system will also apply to the n-th order system. For the general first-order plant, the optimization problem reduces to the minimization of

$$J = \tfrac{1}{2}S\bar{x}^2(t_f) + S\bar{x}(t_f)M(t_f) + \tfrac{1}{2}SP(t_f) \tag{2.17}$$

$$+ \int_{t_o}^{t_f} (\tfrac{1}{2}Q_1\bar{x}^2 + Q_1\bar{x}M + \tfrac{1}{2}Q_1P + \tfrac{1}{2}R_1u^2)dt$$

subject to

$$\dot{\bar{x}} = f(\bar{x},u,t) \tag{2.18}$$

$$\dot{M} = F_xM + \tfrac{1}{2}F_{xx}P \tag{2.19}$$

$$\dot{P} = 2F_xP + Q_w \tag{2.20}$$

where

$$F_x = \frac{\partial f}{\partial x}\bigg|_{x=\bar{x}} \quad \text{and} \quad F_{xx} = \frac{\partial^2 f}{\partial x^2}\bigg|_{x=\bar{x}}$$

All the terms in (2.17)-(2.20) are scalars. Adjoining the constraints to the performance index by Lagrange multipliers $\bar{\lambda}$, $\lambda_M$, and $\lambda_P$, the Hamiltonian is formed as

$$H = \tfrac{1}{2}Q_1\bar{x}^2 + Q_1\bar{x}M + \tfrac{1}{2}Q_1P + \tfrac{1}{2}R_1u^2 + \bar{\lambda}f(\bar{x},u,t) \tag{2.21}$$

$$+ \lambda_M(F_xM + \tfrac{1}{2}F_{xx}P) + \lambda_P(2F_xP + Q_w)$$

The canonical equations are given by

$$\dot{\bar{x}} = \frac{\partial H}{\partial \bar{\lambda}} \qquad \dot{M} = \frac{\partial H}{\partial \lambda_M} \qquad \dot{P} = \frac{\partial H}{\partial \lambda_P}$$

$$-\dot{\bar{\lambda}} = \frac{\partial H}{\partial \bar{x}} \qquad -\dot{\lambda}_M = \frac{\partial H}{\partial M} \qquad -\dot{\lambda}_P = \frac{\partial H}{\partial P}$$

and $$\frac{\partial H}{\partial u} = 0$$

Applying the canonical equations to (2.21), the following differential equations are obtained

$$\dot{\bar{x}} = f(\bar{x},u,t) \tag{2.22}$$

$$\dot{M} = F_x M + \tfrac{1}{2}F_{xx}P$$

$$\dot{P} = 2F_x P + Q_w$$

$$-\dot{\bar{\lambda}} = Q_1\bar{x} + Q_1 M + \frac{\partial}{\partial \bar{x}}[\bar{\lambda}f(\bar{x},u,t) + \lambda_M(F_x M + \tfrac{1}{2}F_{xx}P) + 2\lambda_P F_x P]$$

$$-\dot{\lambda}_M = Q_1\bar{x} + \lambda_M F_x$$

$$-\dot{\lambda}_P = \tfrac{1}{2}Q_1 + \tfrac{1}{2}\lambda_M F_{xx} + 2\lambda_P F_x$$

The equation for the optimality of u is

$$R_1 u + \frac{\partial}{\partial u}[\bar{\lambda}f(\bar{x},u,t) + \lambda_M(F_x M + \tfrac{1}{2}F_{xx}P) + 2\lambda_P F_x P] = 0 \tag{2.23}$$

The boundary conditions for the adjoint equations are

$$\bar{\lambda}(t_f) = \frac{\partial}{\partial \bar{x}}\left[\frac{1}{2}S\bar{x}^2 + S\bar{x}M + \frac{1}{2}SP\right]\Bigg|_{t=t_f}$$

$$\lambda_M(t_f) = \frac{\partial}{\partial M}\left[\frac{1}{2}S\bar{x}^2 + S\bar{x}M + \frac{1}{2}SP\right]\Bigg|_{t=t_f}$$

$$\lambda_P(t_f) = \frac{\partial}{\partial P}\left[\frac{1}{2}S\bar{x}^2 + S\bar{x}M + \frac{1}{2}SP\right]\Bigg|_{t=t_f}$$

The total set of boundary conditions become, after simplification,

$$\begin{aligned} \bar{x}(t_o) &= x_o & \bar{\lambda}(t_f) &= S\bar{x} + SM(t_f) \\ M(t_o) &= 0 & \lambda_M(t_f) &= S\bar{x}(t_f) \\ P(t_o) &= 0 & \lambda_P(t_f) &= \frac{1}{2}S \end{aligned} \qquad (2.24)$$

Some comments on the equations in (2.22) and (2.23) are now appropriate. Although these equations are for the first-order system, the following comments also apply to the general nth-order system.

1. When the system is nonlinear in either the control or state, all of the resulting equations in (2.22) and (2.23) are coupled and determine an open-loop control different from the deterministic optimal control. Therefore, a better control than the deterministic optimal control always exists.

2. Whenever the plant is linear, then it is obvious that $M = 0$. The $\dot{\bar{x}}$ and $\dot{\bar{\lambda}}$ equations are then uncoupled from the other equations, and the problem reduces to the determination of the deterministic optimal control. Therefore, the deterministic optimal control is always the best open-loop control for a linear system.

<u>A Numerical Example</u>

To determine what improvements could be realized by the method just introduced, a particular nonlinear first-order example was worked. The problem may be stated as the minimization of

$$J = E\left[\int_0^{1.5}(x^2 + u^2)dt\right] \tag{2.25}$$

subject to the nonlinear stochastic equation

$$\dot{x} = -.5x + .25x^3 - .035x^5 + u + w \tag{2.26}$$

The system nonlinearity is shown in Figure 2. An initial condition of $x_o = 3.0$ was chosen. So that comparisons could be made between the performance obtained by using the trajectory optimized control and that obtained by using the deterministic optimal control, the deterministic optimal control problem obtained by setting w equal to zero in (2.26)

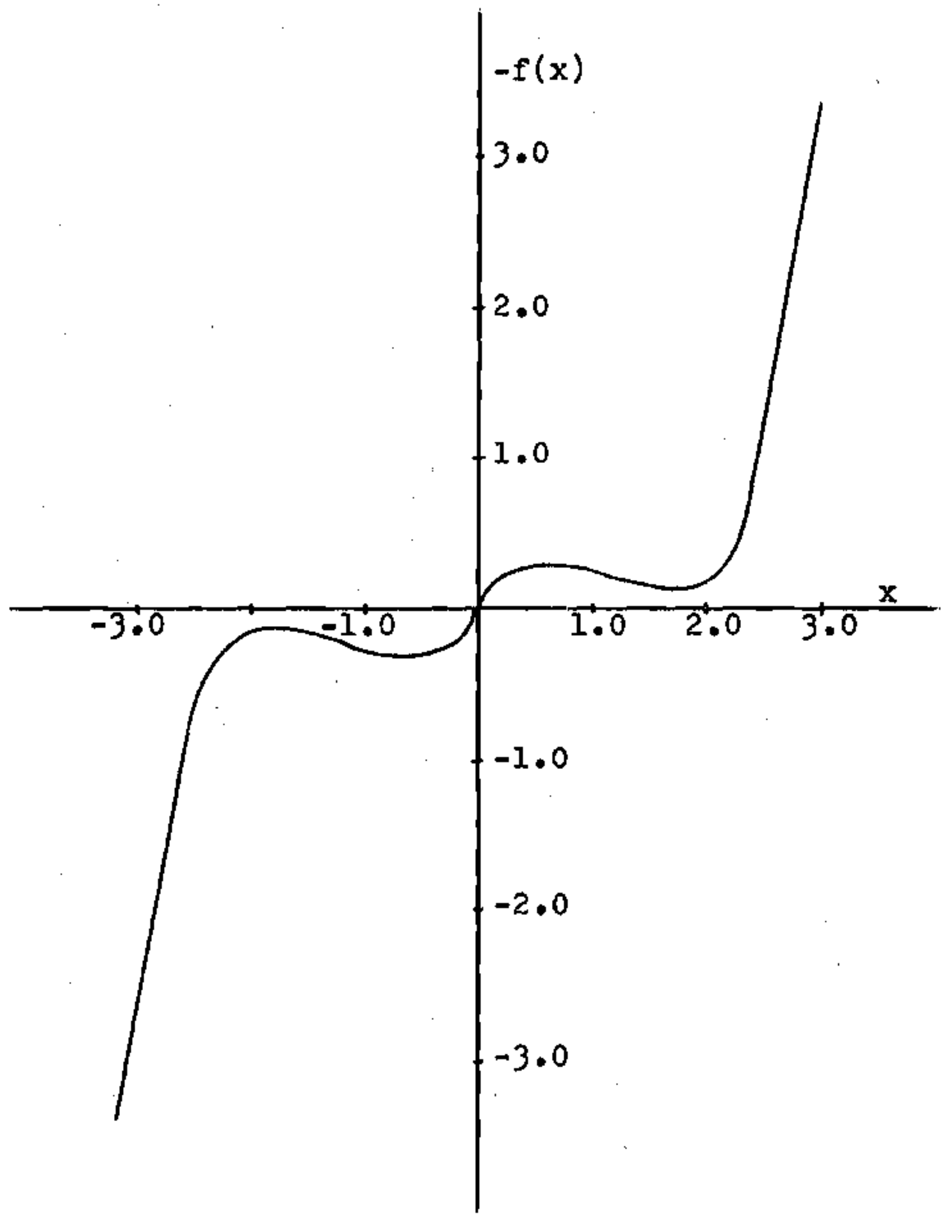


Figure 2. The Nonlinearity for the Open-Loop Example

was also considered. The deterministic problem reduced to the minimization of

$$J_d = \int_0^{1.5} (x^2 + u^2)dt \tag{2.27}$$

subject to

$$\dot{x} = -.5x + .25x^3 - .035x^5 + u \tag{2.28}$$

The two-point boundary value problem posed by the trajectory optimization method applied to (2.25) and (2.26) and the problem posed by (2.27) and (2.28) were solved on the digital computer by using the gradient method* [47].

In Figure 3, a plot is given of the deterministic optimal control along with the controls obtained by trajectory optimization for various values of $Q_w$. Also, a plot is given in Figure 4 of the nominal trajectories associated with these nominal controls. A marked difference in the optimal deterministic nominal and the nominals

---

*In using the gradient method, one should be aware of the possibility of the existence of relative minima. In the numerical example, the plant is linear in the control so that $H_{uu}=2.0$ for all t. This observation coupled with the fact that no conjugate points exist guarantees sufficient conditions for a strong minimum where the necessary conditions for a minimum are met [39]. However, for general nonlinear systems it is recommended that one use several initial starting controls so that all relative minima are detected and the absolute minimum is found.

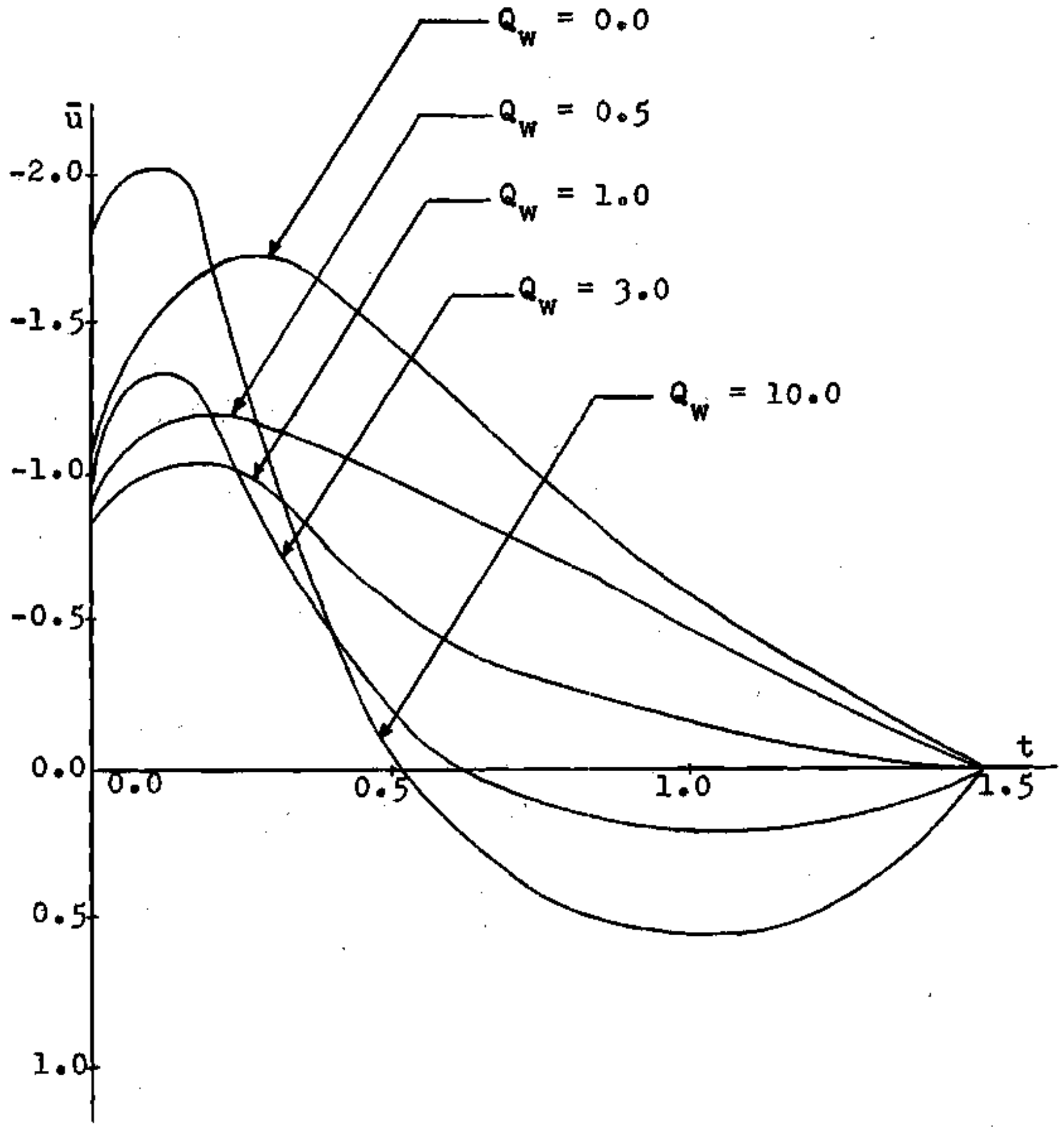


Figure 3. Stochastic Controls for the Open-Loop Problem

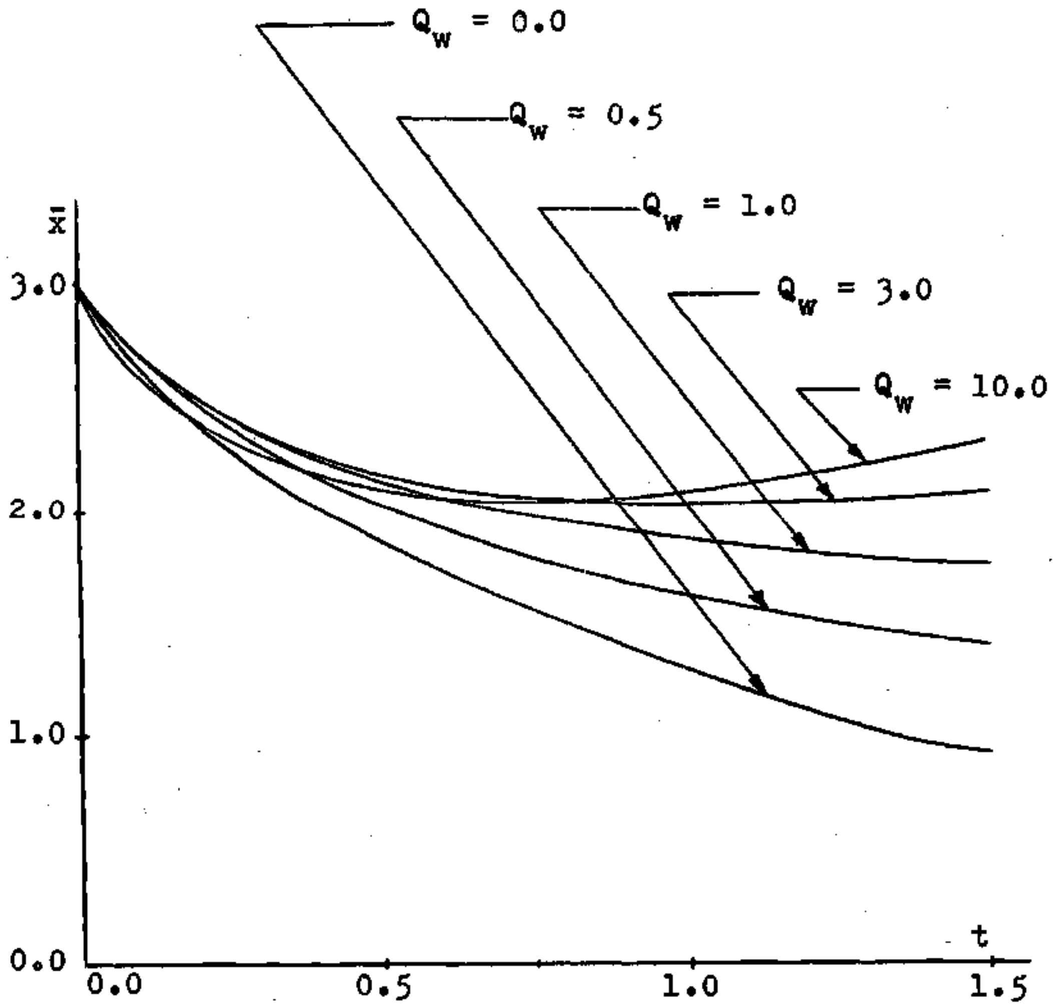


Figure 4. Stochastic Nominal Trajectories for the Open-Loop Problem

obtained by trajectory optimization is evident from the figures.

A Monte Carlo simulation using 100 runs was performed to compare the performance obtained with trajectory optimization to the performance obtained by using the deterministic optimal open-loop control. It was observed that as the number of runs was increased to 125, J for both of the methods increased approximately 0.5 percent. Consequently, it was concluded that 100 runs was probably sufficient to obtain an accurate comparison of the two methods.

In simulating the continuous white noise process w on the digital computer a random number generator was used which produced independent zero-mean samples from a given distribution. If $Q_{wd}$ denotes the variance of these samples, and T is the computer step size, then one has [48]

$$E[wd(t)wd(\tau)] = \begin{cases} Q_{wd}\left(1 - \frac{|\tau-t|}{T}\right) & \text{for } |\tau-t| < T \\ 0 & \text{otherwise} \end{cases} \qquad (2.29)$$

To guarantee that (2.29) will approach (1.2) as T approaches zero and the discrete process approaches a continuous process, one must have

$$Q_{wd} = \frac{Q_w}{T}$$

The probability density function of the discrete samples was chosen according to the nongaussian density function given by

$$p_{wd}(wd) = \begin{cases} 6.5\sqrt{Q_{wd}}(wd)^{12} & \text{for } -\sqrt{Q_{wd}}<wd<\sqrt{Q_{wd}} \\ 0 & \text{otherwise} \end{cases} \quad (2.30)$$

In Figure 5, a plot is given of the performance indices obtained using the trajectory optimized control as well as the deterministic optimal control. In Figure 6, a plot is shown of the percent reduction in J versus $Q_w$. It is evident that improvement was obtained by trajectory optimization for small to moderate values of $Q_w$. At extremely low values of $Q_w$, the method yielded no improvement, which could probably be attributed to approximations in the method. At these extremely low values of $Q_w$, it would appear that even with more accurate approximations, only small improvements over the deterministic optimal control could be realized. However, as shown in Figure 6, for a fairly wide range of $Q_w$, up to five percent reduction in J may be obtained by the method of trajectory optimization.

## Conclusions

This chapter has introduced the method of trajectory optimization by considering the open-loop nonlinear stochastic

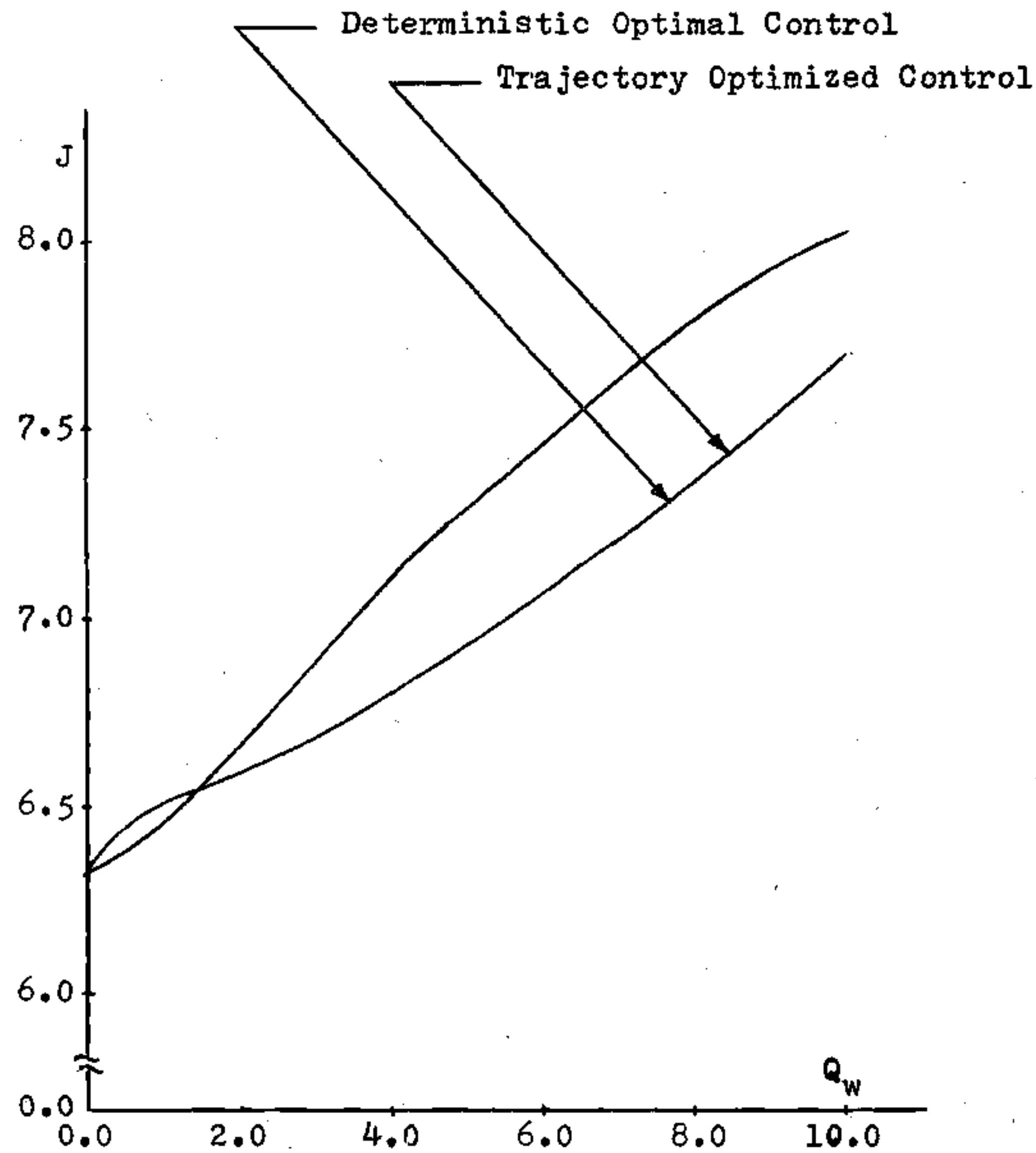


Figure 5. Performance Index Versus $Q_w$ for the Open-Loop Problem

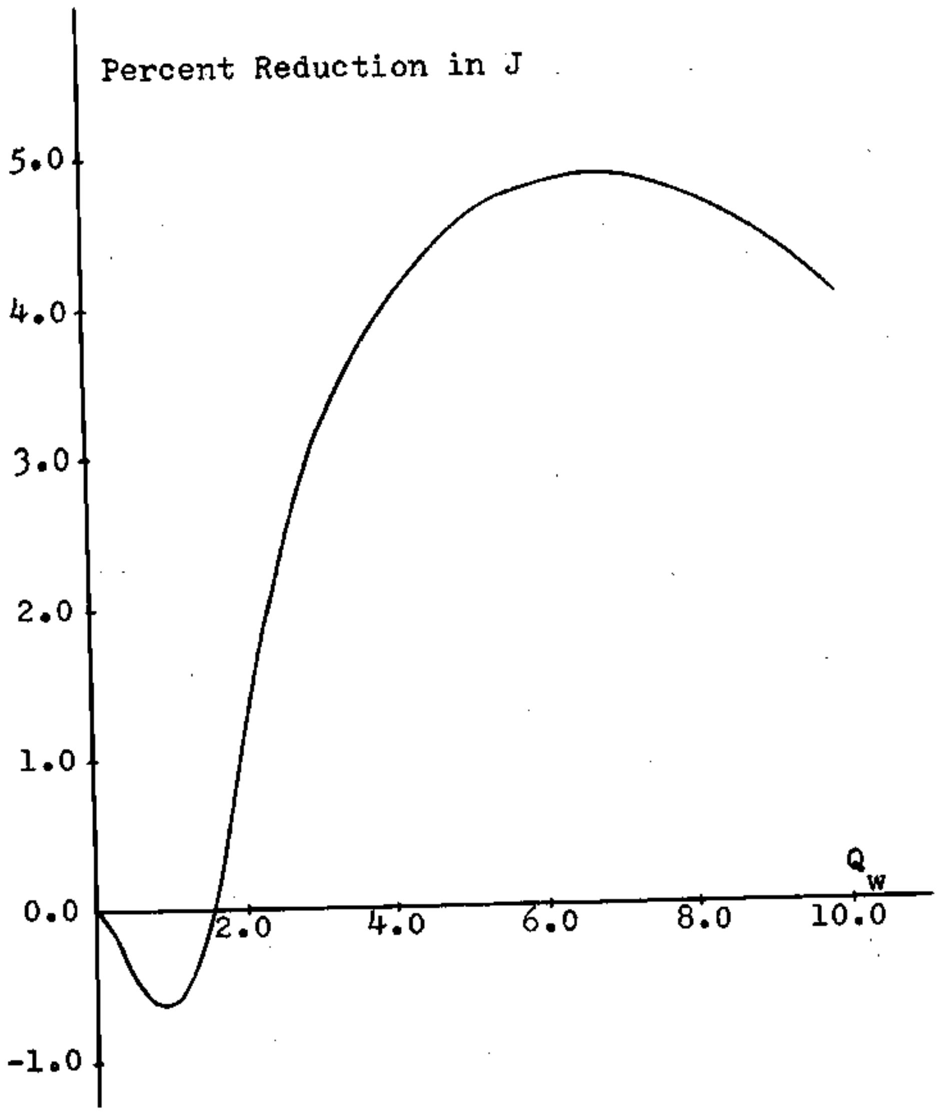


Figure 6. Percent Reduction in J Versus $Q_w$ for the Open-Loop Problem

system. The stochastic optimization problem was reduced to a deterministic calculus of variations problem. It was demonstrated that the performance index is very sensitive to the choice of nominal trajectory and that significant increase in performance is realizable by employing the trajectory optimization method. In implementing trajectory optimization, a deterministic optimization problem of higher order than that of the system must be solved. Therefore, greater accuracy may be obtained at the expense of solving an optimization problem of higher dimension.

# CHAPTER III

# THE CLOSED-LOOP PROBLEM WITH ZERO MEASUREMENT NOISE

## Introduction

In this chapter, the extension of the method of trajectory optimization is described for the case where a closed-loop control is required. It is assumed in the closed-loop case that exact knowledge of the state is available for feedback purposes. Special emphasis is given to the manner in which the Riccati controller is modified by incorporating a Riccati equation into the optimization problem. A nonlinear numerical example is presented to illustrate trajectory optimization as applied to the closed-loop problem.

## Mathematical Development

In the closed-loop problem it is assumed again that the state may be written as in (2.1). In addition it is assumed that the total control may be written as

$$u = \bar{u} + \delta u \tag{3.1}$$

where $\bar{u}$ is the deterministic control schedule to be optimized along with $\bar{x}$, and $\delta u$ is a corrective perturbation feedback control term. Substituting (2.1) and (3.1) into the performance index (1.5) yields

$$J = \frac{1}{2}\bar{x}^T(t_f)S\bar{x}(t_f) + \bar{x}^T(t_f)SE[\delta x(t_f)] \quad (3.2)$$

$$+ \frac{1}{2} \text{ trace } SE[\delta x(t_f)\delta x^T(t_f)]$$

$$+ \int_{t_o}^{t_f} (\frac{1}{2}\bar{x}^T Q_1 \bar{x} + \bar{x}^T Q_1 E(\delta x) + \frac{1}{2} \text{ trace } Q_1 E[\delta x \delta x^T]$$

$$+ \frac{1}{2}\bar{u}^T R_1 \bar{u} + \bar{u}^T R_1 E(\delta u)$$

$$+ \frac{1}{2} \text{ trace } R_1 E[\delta u \delta u^T])dt$$

A perturbation feedback control of the form

$$\delta u = -K_u \delta x \quad (3.3)$$

is introduced. The term $K_u$ is an m by n time-varying matrix.

A problem in the calculus of variations is formulated with the constraint that for linear plants the perturbation controller will reduce to the regular Riccati controller. When the method is applied to nonlinear systems, the nominal trajectory will be optimized along with the elements of $K_u$. A symmetric matrix $P_u$ is introduced by the relationship

$$K_u = R_1^{-1} G^T P_u \quad (3.4)$$

where

$$G = \left(\frac{\partial f}{\partial u}\right)^T \Bigg|_{\substack{x=\bar{x} \\ u=\bar{u}}}$$

Substituting (3.3) and (3.4) into (3.2) as well as the expressions for $E(\delta x)$ and $E(\delta x \delta x^T)$, as given by M and P, respectively, one has

$$J = \tfrac{1}{2}\bar{x}^T(t_f)S\bar{x}(t_f) + \bar{x}^T(t_f)SM(t_f) + \tfrac{1}{2}\,\text{trace}\,SP(t_f) \tag{3.5}$$

$$+ \int_{t_o}^{t_f} (\tfrac{1}{2}\bar{x}^T Q_1 \bar{x} + \bar{x}^T Q_1 M + \tfrac{1}{2}\,\text{trace}\,Q_1 P + \tfrac{1}{2}\bar{u}^T R_1 \bar{u}$$

$$-\bar{u}^T G^T P_u M + \tfrac{1}{2}\,\text{trace}\,G^T P_u P P_u G R_1^{-1})dt$$

The equation (3.5) represents a deterministic performance index. The next step in the application of trajectory optimization is to find differential equations for $\bar{x}$, M, P, and $P_u$. The first constraint is obtained by requiring, as in the open-loop problem, that the nominal trajectory $\bar{x}$ satisfy the plant equation with no noise. i.e.

$$\dot{\bar{x}} = f(\bar{x},\bar{u},t) \tag{3.6}$$

The perturbation equation for $\delta\dot{x}$ is expanded to first order to yield

$$\delta\dot{x} = F\delta x + G\delta u + w \tag{3.7}$$

where

$$F = \left(\frac{\partial x}{\partial x}\right)^T \Bigg|_{\substack{x=\bar{x}\\ u=\bar{u}}}$$

Substituting for $\delta u$ in (3.7) one obtains

$$\delta\dot{x} = F\delta x - GR_1^{-1}G^TP_u\delta x + w \tag{3.8}$$

Since (3.8) is linear, the matrix P satisfies the equation

$$\dot{P} = (F - GR_1^{-1}G^TP_u)P + P(F - GR_1^{-1}G^TP_u)^T + Q_w \tag{3.9}$$

For the $\dot{M}$ equation, $\delta\dot{x}$ is expanded to second-order. Taking expectations, one obtains

$$\begin{aligned}\dot{M}_j = &\sum_{i=1}^{n} F_{ji}M_i - \sum_{i=1}^{m}\sum_{r=1}^{n} G_{ji}[R_1^{-1}G^TP_u]_{ir}M_r \\ &+ \frac{1}{2}\sum_{k=1}^{n}\sum_{i=1}^{n} F_{jik}P_{ik} - \sum_{k=1}^{n}\sum_{i=1}^{m}\sum_{r=1}^{n} T_{jik}[R_1^{-1}G^TP_u]_{ir}P_{kr} \\ &+ \frac{1}{2}\sum_{k=1}^{m}\sum_{i=1}^{m}\sum_{r=1}^{n}\sum_{q=1}^{n} G_{jik}[R_1^{-1}G^TP_u]_{ir}[R_1^{-1}G^TP_u]_{kq}P_{rq}\end{aligned} \tag{3.10}$$

for $j = 1,2,\ldots\ldots,n$

where

$$F_{ji} = \frac{\partial f_j}{\partial x_i}\bigg|_{\substack{x=\bar{x}\\u=\bar{u}}} \quad G_{ji} = \frac{\partial f_j}{\partial u_i}\bigg|_{\substack{x=\bar{x}\\u=\bar{u}}} \quad F_{jik} = \frac{\partial^2 f_j}{\partial x_i \partial x_k}\bigg|_{\substack{x=\bar{x}\\u=\bar{u}}}$$

$$G_{jik} = \frac{\partial^2 f_j}{\partial u_i \partial u_k}\bigg|_{\substack{x=\bar{x}\\u=\bar{u}}} \quad \text{and} \quad T_{jik} = \frac{\partial^2 f_j}{\partial u_i \partial x_k}\bigg|_{\substack{x=\bar{x}\\u=\bar{u}}}$$

The $P_u$ matrix is constrained to satisfy the Riccati equation

$$\dot{P}_u = -P_uF - F^TP_u + P_uGR_1^{-1}G^TP_u - Q_1 \qquad (3.11)$$

If the system (1.1) is linear, F and G are independent of $\bar{x}$ and $\bar{u}$, and (3.11) reduces to the regular Riccati equation. However, for the given nonlinear system, (3.11) is influenced by both the nominal control and nominal trajectory. The Riccati equation in (3.11) is used as a constraint in the optimization problem to optimize the feedback control matrix $K_u$ along with the nominal control $\bar{u}$ and the nominal trajectory $\bar{x}$. Therefore, the nonlinear stochastic control problem is reduced to a deterministic control problem in which (3.5) is minimized subject to the $n^2 + 3n$ constraints given by (3.6), (3.9), (3.10), and (3.11). Although the approach is approximate, the method does consider stochastic effects up to second-order in the shaping of the nominal trajectory, nominal control, and the perturbation feedback controller. The structure of the system resulting from the use of this method is shown in Figure 7.

## Application to the General First-order System

The closed-loop trajectory optimization procedure described in the previous section is now applied to a general first-order system. The Hamiltonian is formed by adjoining the constraints to the performance index through Lagrange multipliers $\bar{\lambda}$, $\lambda_M$, $\lambda_P$, and $\lambda_{P_u}$. The Hamiltonian becomes

$$H = \frac{1}{2}Q_1\bar{x}^2 + Q_1M\bar{x} + \frac{1}{2}Q_1P + \frac{1}{2}R_1\bar{u}^2 - G_u\bar{u}P_uM + \frac{1}{2}\,\frac{G_u^2P_u^2P}{R_1} \qquad (3.12)$$

$$+ \bar{\lambda}f(\bar{x},\bar{u},t) + \lambda_P\Big[2\Big(F_x - \frac{G_u^2P_u}{R_1}\Big)P + Q_w\Big]$$

$$+ \lambda_M\Big[F_xM - \frac{G_u^2P_uM}{R_1} + \frac{1}{2}F_{xx}P + \frac{1}{2}\,\frac{G_{uu}G_u^2P_u^2P}{R_1^2} - \frac{F_{xu}G_uP_uP}{R_1}\Big]$$

$$+ \lambda_{P_u}\Big[-2F_xP_u + \frac{G_u^2P_u^2}{R_1} - Q_1\Big]$$

where

$$F_x = \frac{\partial f}{\partial x}\bigg|_{\substack{x=\bar{x}\\ u=\bar{u}}} \qquad F_{xx} = \frac{\partial^2 f}{\partial x^2}\bigg|_{\substack{x=\bar{x}\\ u=\bar{u}}} \qquad F_{xu} = \frac{\partial^2 f}{\partial x\partial u}\bigg|_{\substack{x=\bar{x}\\ u=\bar{u}}}$$

$$G_u = \frac{\partial f}{\partial u}\bigg|_{\substack{x=\bar{x}\\ u=\bar{u}}} \qquad \text{and} \qquad G_{uu} = \frac{\partial^2 f}{\partial u^2}\bigg|_{\substack{x=\bar{x}\\ u=\bar{u}}}$$

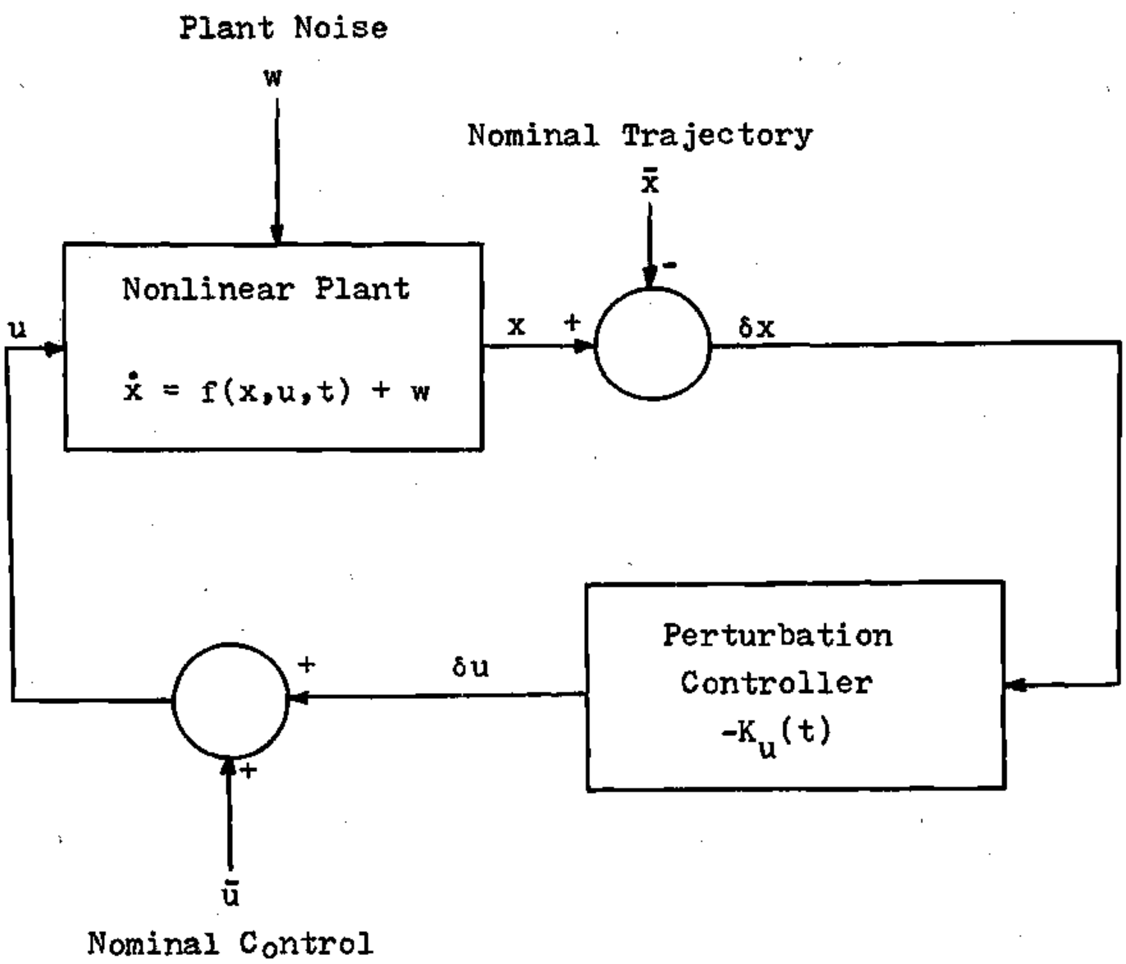


Figure 7. Structure of the Closed-Loop System

The canonical equations are

$$\dot{\bar{x}} = \frac{\partial H}{\partial \bar{\lambda}} \qquad \dot{M} = \frac{\partial H}{\partial \lambda_M} \qquad \dot{P} = \frac{\partial H}{\partial \lambda_P} \qquad \dot{P}_u = \frac{\partial H}{\partial \lambda_{P_u}}$$

$$-\dot{\bar{\lambda}} = \frac{\partial H}{\partial \bar{x}} \qquad -\dot{\lambda}_M = \frac{\partial H}{\partial M} \qquad -\dot{\lambda}_P = \frac{\partial H}{\partial P} \qquad -\dot{\lambda}_{P_u} = \frac{\partial H}{\partial P_u}$$

and

$$\frac{\partial H}{\partial \bar{u}} = 0$$

Applying the canonical equations to (3.12), one obtains

$$\dot{\bar{x}} = f(\bar{x},\bar{u},t) \tag{3.13}$$

$$\dot{M} = F_x M - \frac{G_u^2 P_u M}{R_1} + \frac{1}{2}F_{xx}P + \frac{1}{2}\,\frac{G_{uu}G_u^2 P_u^2 P}{R_1^2} - \frac{F_{xu}G_u P_u P}{R_1}$$

$$\dot{P} = 2(F_x - \frac{G_u^2 P_u}{R_1})P + Q_w$$

$$\dot{P}_u = -2F_x P_u + \frac{G_u^2 P_u^2}{R_1} - Q_1$$

$$-\dot{\bar{\lambda}} = Q_1\bar{x} + Q_1 M + \frac{\partial}{\partial \bar{x}}[-G_u \bar{u} P_u M + \frac{1}{2}\,\frac{G_u^2 P_u^2 P}{R_1} + \bar{\lambda} f(\bar{x},\bar{u},t)$$
$$+ \lambda_M(F_x M - \frac{G_u^2 P_u M}{R_1} + \frac{1}{2}\,\frac{G_{uu}G_u^2 P_u^2 P}{R_1^2} - \frac{F_{xu}G_u P_u P}{R_1} + \frac{1}{2}F_{xx}P)$$
$$+ 2\lambda_P\frac{(F_x - G_u^2 P_u)P}{R_1} + \lambda_{P_u}(-2F_x P_u + \frac{G_u^2 P_u^2}{R_1})]$$

$$-\dot{\lambda}_M = Q_1\bar{x} - G_u\bar{u}P_u + \lambda_M\left(F_x - \frac{G_u^2P_u}{R_1}\right)$$

$$-\dot{\lambda}_P = \tfrac{1}{2}Q_1 + \tfrac{1}{2}\,\frac{G_u^2P_u^2}{R_1} + 2\lambda_P\left(F_x - \frac{G_u^2P_u}{R_1}\right)$$

$$+ \lambda_M\left(\tfrac{1}{2}F_{xx} + \tfrac{1}{2}\,\frac{G_{uu}G_u^2P_u^2}{R_1^2} - \frac{F_{xu}G_uP_u}{R_1}\right)$$

$$-\dot{\lambda}_{P_u} = -G_u\bar{u}M + \frac{G_u^2P_uP}{R_1} - \frac{2\lambda_PG_u^2P}{R_1} + \lambda_M\left(\frac{-G_u^2M}{R_1} + \frac{G_{uu}G_u^2P_uP}{R_1^2} - \frac{F_{xu}G_uP}{R_1}\right)$$

$$+ \lambda_{P_u}\left(-2F_x + \frac{2G_u^2P_u}{R_1}\right)$$

$$R_1\bar{u} - G_uP_uM + \frac{\partial}{\partial\bar{u}}\Big[\bar{\lambda}f(\bar{x},\bar{u},t) + 2\lambda_P\left(F_x - \frac{G_u^2P_u}{R_1}\right)P + \tfrac{1}{2}\,\frac{G_u^2P_u^2P}{R_1}$$

$$+ \lambda_M\left(F_xM - \frac{G_u^2P_uM}{R_1} + \tfrac{1}{2}F_{xx}P + \tfrac{1}{2}\,\frac{G_{uu}G_u^2P_u^2P}{R_1^2} - \frac{F_{xu}G_uP_uP}{R_1}\right)$$

$$+ \lambda_{P_u}\left(-2F_xP_u + \frac{G_u^2P_u^2}{R_1}\right)\Big] = 0$$

In addition to the boundary conditions in (2.24), the boundary conditions for $P_u$ and $\lambda_{P_u}$ are

$$P_u(t_f) = S \quad \text{and} \quad \lambda_{P_u}(t_o) = 0 \tag{3.14}$$

Note that the boundary condition on $P_u$ is identical to that obtained from the regular Riccati equation. The boundary condition for $\lambda_{p_u}$ follows directly from the application of calculus of variations theory to the optimization problem. When the plant is nonlinear, all the equations in (3.13) are coupled and determine a nominal trajectory, nominal control, and perturbation feedback matrix different from those obtained by simple linearization about the deterministic optimal trajectory. However, for linear plants, one obtains the deterministic optimal trajectory and Riccati controller. An interesting feature in this trajectory optimization method is that all calculations are performed off-line. The nominal control, nominal trajectory, and feedback matrices are stored in the digital computer for on-line operation.

## Another Approximate Perturbation Feedback Method

A widely used approximate method for the nonlinear closed-loop problem is to linearize the system about the deterministic optimal trajectory [39] to yield

$$\delta\dot{x} = F\delta x + G\delta u + w \tag{3.15}$$

where F and G are evaluated at the deterministic optimal $\bar{x}$ and $\bar{u}$. A perturbation feedback control as in (3.3) is employed such that the perturbation performance index given by

$$J_\delta = \tfrac{1}{2}\delta x(t_f)S'\delta x^T(t_f) + \int_{t_o}^{t_f}(\tfrac{1}{2}\delta x Q_1'\delta x^T + \tfrac{1}{2}\delta u R_1'\delta u^T)dt \tag{3.16}$$

is minimized. Sage [47] has noted that, in general, the choice of the weighting matrices $S'$, $Q_1'$, and $R_1'$ is somewhat arbitrary. Often the matrices are taken to be

$$S' = S \quad Q_1' = Q_1 \quad R_1' = R_1 \tag{3.17}$$

as in the original performance index. However, Breakwell, Speyer, and Bryson [49] have done some work on approximations for this problem in which they attempted to minimize the second variation of J. They obtained an approximate criterion for choosing the weighting matrices $S'$, $Q_1'$, and $R_1'$. If the control in the nonlinear plant is not state-dependent, these weighting matrices are chosen as [49]

$$S' = S \quad Q_1' = \left.\frac{\partial^2 H'}{\partial x^2}\right|_{\substack{x=\bar{x}\\u=\bar{u}}} \quad R_1' = \left.\frac{\partial^2 H'}{\partial u^2}\right|_{\substack{x=\bar{x}\\u=\bar{u}}} \tag{3.18}$$

where $H'$ is the Hamiltonian obtained when the deterministic optimization problem is solved. The perturbation feedback matrix becomes

$$K_u = -[R_1']^{-1}G^TP_u \tag{3.19}$$

and $P_u$ satisfies the Riccati equation

$$\dot{P}_u = -P_uF - F^TP_u + P_uG[R_1']^{-1}G^TP_u - Q_1' \tag{3.20}$$

with

$$P_u(t_f) = S'$$

In the development of this second-variation method, stochastic disturbances are not considered and, therefore, the value of the method in handling stochastic systems is not known. In the next section, the second-variation method is compared in a particular example to the method of trajectory optimization as well as to the method in which the weighting matrices are selected according to (3.17).

## A Numerical Example

In this section, a first-order numerical example is presented to show the improvement that can be obtained when the trajectory optimization method is utilized. The equations in (3.13) are applied to the nonlinear plant in (2.26) and the performance index in (2.25). The second-variation method is also applied to the system with weighting matrices in (3.18) and Riccati controller determined from (3.20). The deterministic optimal trajectory is employed in the second-variation method. The perturbation feedback control problem is also solved with the deterministic optimal trajectory with weighting matrices given in (3.17) and Riccati controller determined from (3.20).

In Figure 8, a plot is given of the deterministic optimal control as well as several stochastic nominal controls determined by trajectory optimization. In Figure 9, the nominal trajectories corresponding to these nominal controls are plotted. Again, a marked difference in the deterministic optimal nominal and the stochastic nominals is observed. The perturbation feedback gains corresponding to these nominal trajectories are given in Figure 10. The feedback gains corresponding to the deterministic optimal trajectory are indicated for the second-variation method as well as for the perturbation method with weighting matrices as in (3.17). It is noted that the second-variation method produces negative feedback gains along a portion of the trajectory.

A Monte Carlo simulation was performed to compare the performance obtained with trajectory optimization to the performance obtained by the two methods previously indicated which utilize the deterministic optimal trajectory. The density function of the discrete random number generator samples wd was again taken as in (2.30). In Figure 11, the performance indices are plotted versus $Q_w$ for these three methods. At low values of plant noise, the deterministic optimal trajectory method utilizing $Q_1' = Q_1$ yielded slightly better performance than the trajectory optimization method. The performance of the system utilizing the second-variation method was considerably poorer than the performance obtained when either trajectory optimization or the method with $Q_1' = Q_1$ was employed. At moderate to large values of $Q_w$, the

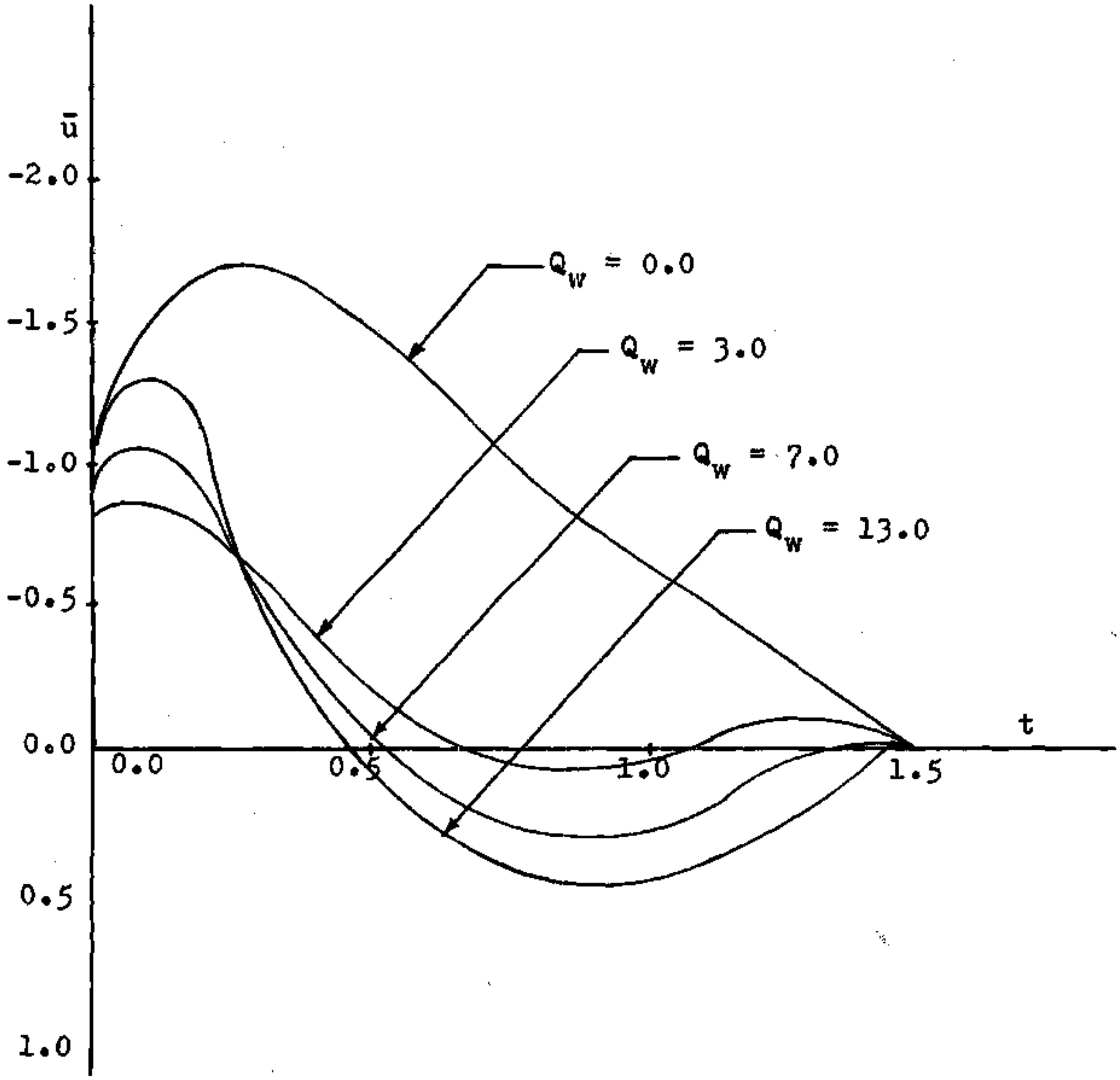


Figure 8. Stochastic Nominal Controls for the Closed-Loop Problem

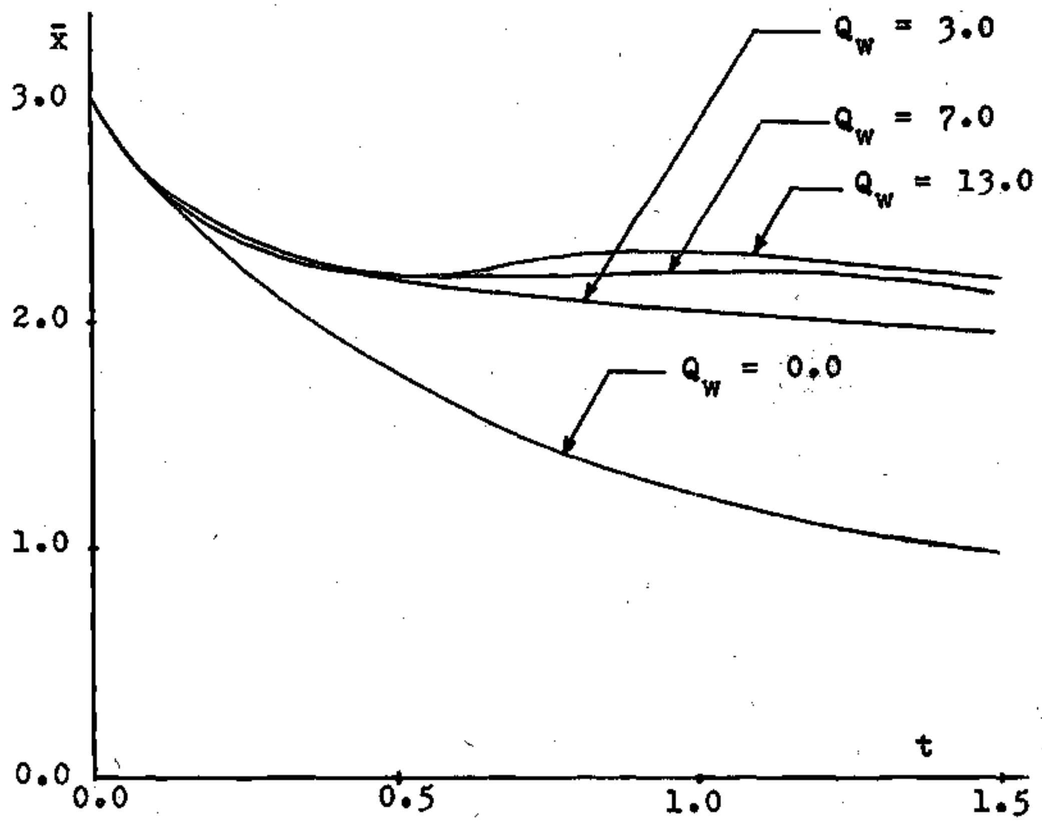


Figure 9. Stochastic Nominal Trajectories for the Closed-Loop Problem

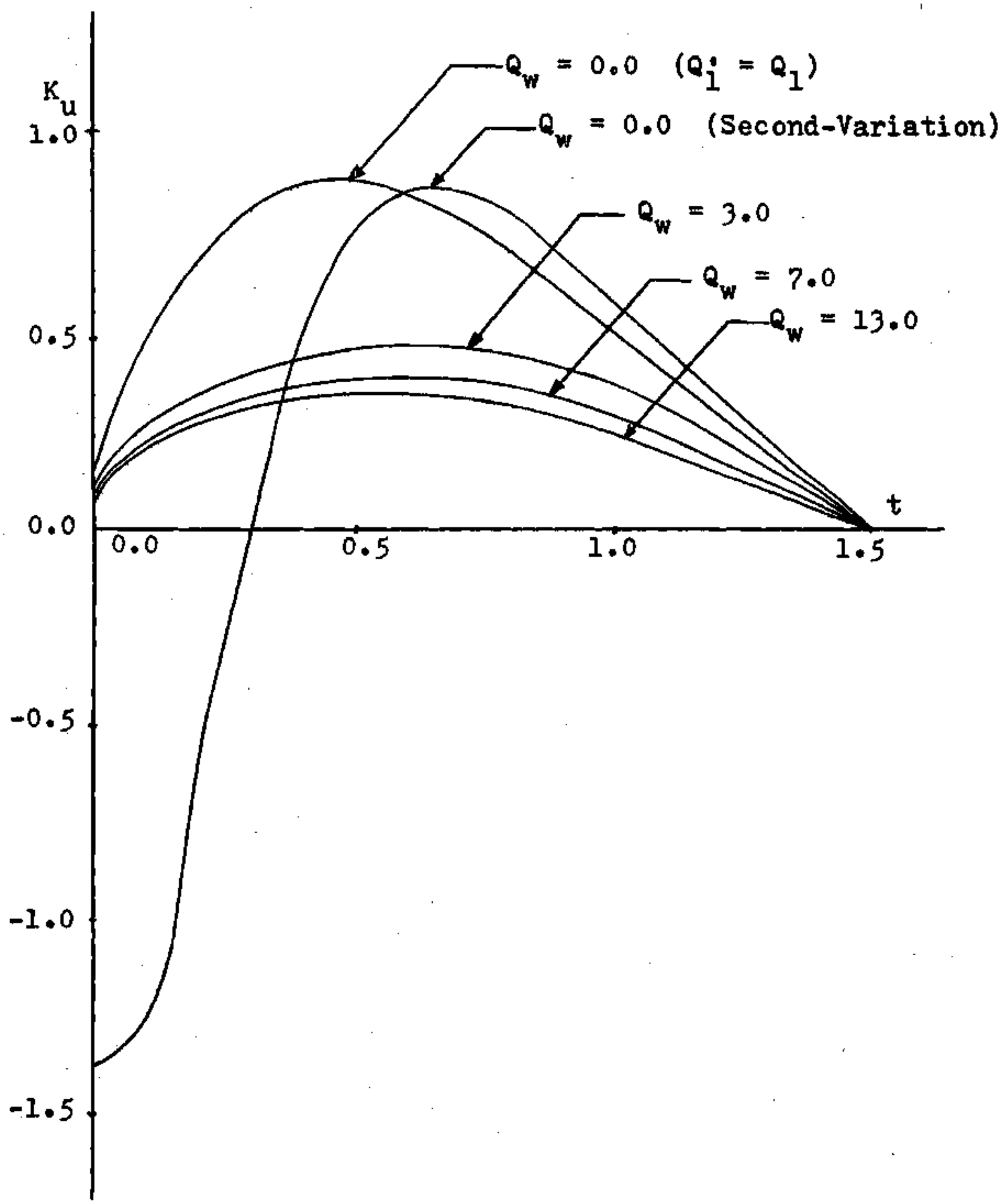


Figure 10. Stochastic Perturbation Feedback Gains for the Closed-Loop System

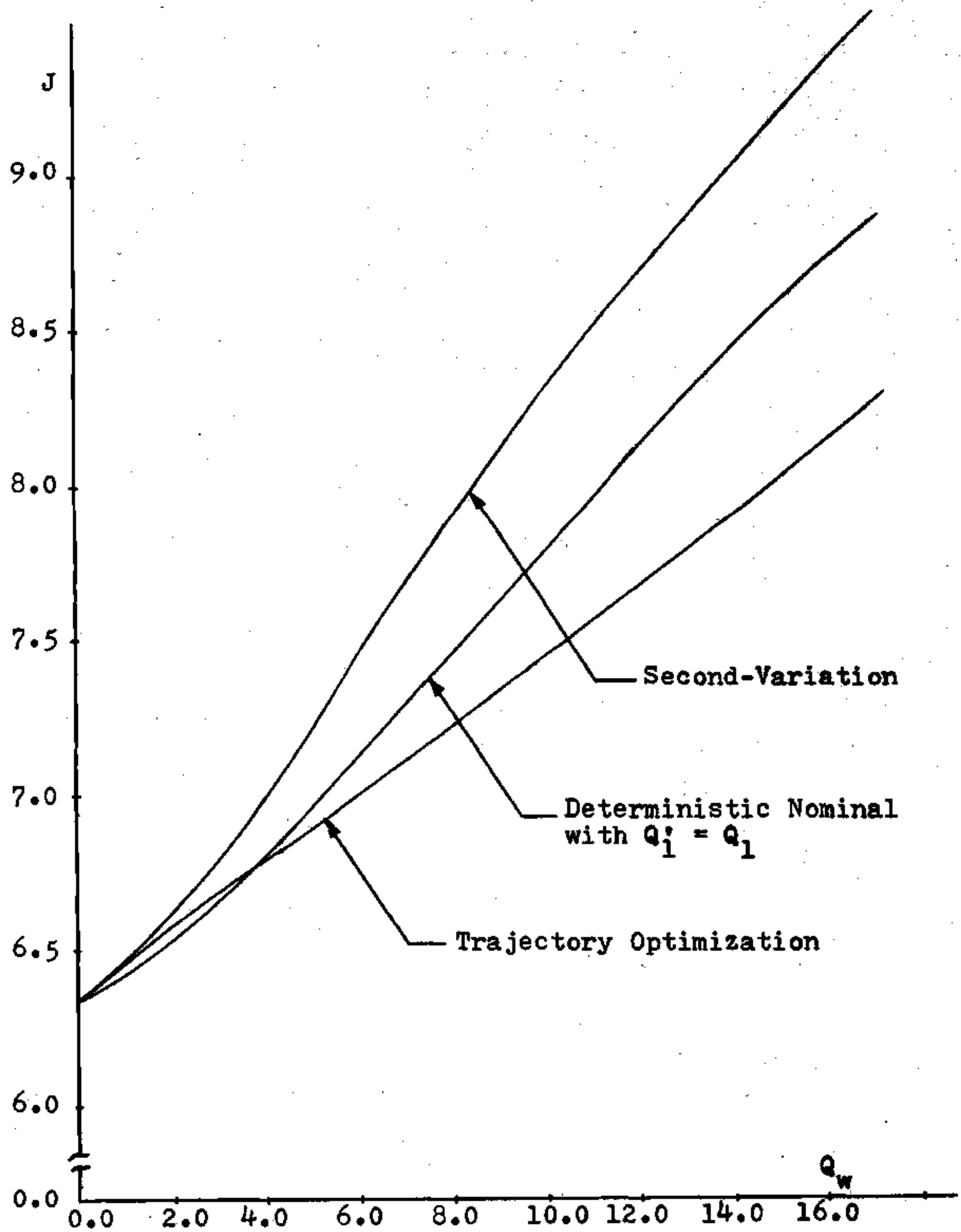


Figure 11. Performance Index Versus $Q_w$ for the Closed-Loop Problem

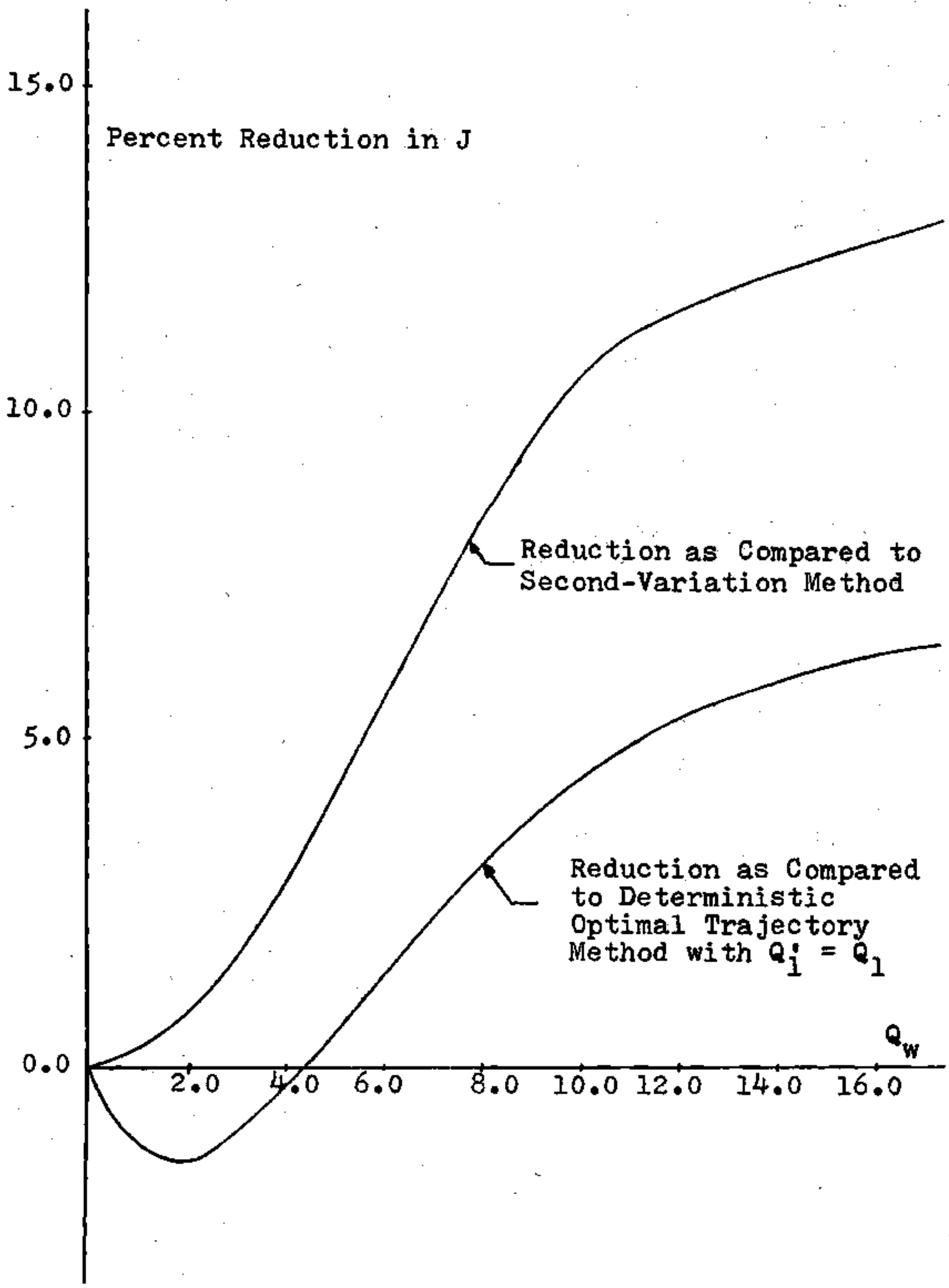


Figure 12. Percent Reduction in J Versus $Q_w$ for the Closed-Loop Problem

trajectory optimization method was clearly superior to the other two methods. Up to 6.5 percent reduction in J was realized over the method in which $Q_1' = Q_1$ was used as the weighting matrix. Furthermore, up to 14.0 percent was obtained over the method which utilized the second-variation. A plot of percent reduction in J versus $Q_w$ is shown in Figure 12.

## Conclusions

It has been demonstrated in this chapter that trajectory optimization could be extended to the closed-loop problem. The optimization problem was formulated to include a Riccati controller in the constraint equations. A significant improvement in performance was obtained using trajectory optimization as compared to the performance obtained by linearizing about the deterministic optimal trajectory.

# CHAPTER IV

# TRAJECTORY OPTIMIZATION FOR ESTIMATION AND CONTROL WITH LINEAR FILTERING

## Introduction

This chapter describes the application of trajectory optimization to the nonlinear combined estimation and control problem. In the development of the method, a linear perturbation estimator is employed. Although the forms of the filter and controller are assumed, the parameters for the filter and controller are simultaneously chosen with the nominal trajectory and nominal control to minimize the performance index. Since a general separation theorem does not exist for nonlinear systems, this simultaneous optimization of system parameters with the nominal trajectory appears to be a realistic approach.

## Mathematical Development

The objective in the combined estimation and control problem is to minimize (1.5) subject to the constraints given in (1.1)-(1.4). In the development of the trajectory optimization solution for the problem, it is assumed that the state and control may be written as in (2.1) and (3.1). A perturbation controller of the form

$$\delta u = -K_u \delta \hat{x} \tag{4.1}$$

is assumed. In (4.1), $\delta\hat{x}$ is the estimate of the deviation of the state from the nominal trajectory. The gain matrix $K_u$ is chosen in such a way that for linear systems the controller reduces to the regular Riccati controller and for nonlinear systems is influenced by the choice of nominal trajectory. It is assumed that $K_u$ satisfies (3.4) and (3.11). For the estimator, a linear filter is chosen as

$$\delta\dot{\hat{x}} = F\delta\hat{x} + G\delta u + K_e(\delta z - H_e\delta\hat{x}) \tag{4.2}$$

where

$$H_e = \left|\frac{\partial h}{\partial x}\right|^T \Bigg|_{\substack{x=\bar{x}\\ u=\bar{u}}}$$

and

$$\delta z = z - h(\bar{x}) \tag{4.3}$$

Equation (4.2) is simply the Kalman filter with the time-varying filter gain $K_e$ which is chosen along with $K_u$ in the trajectory optimization problem. When (2.1), (3.1), and (4.1) are substituted into the performance index (1.5), one obtains the expression for J given on the next page.

$$J = \tfrac{1}{2}\bar{x}^T(t_f)S\bar{x}(t_f) + \bar{x}^T(t_f)SE[\delta x(t_f)] \tag{4.4}$$

$$+ \tfrac{1}{2}\,\text{trace}\;SE[\delta x(t_f)\delta x^T(t_f)]$$

$$+\int_{t_o}^{t_f} (\tfrac{1}{2}\bar{x}^T Q_1 \bar{x} + \bar{x}^T Q_1 E(\delta x) + \tfrac{1}{2}\,\text{trace}\;Q_1 E(\delta x \delta x^T) + \tfrac{1}{2}\bar{u}^T R_1 \bar{u}$$

$$-\bar{u}^T R_1 K_u E(\delta\hat{x}) + \tfrac{1}{2}\,\text{trace}\;R_1 K_u E(\delta\hat{x}\delta\hat{x}^T)K_u^T)dt$$

To formulate a calculus of variations problem, constraint equations are found for parameters appearing in the performance index. It is assumed that the filter is approximately unbiased so that

$$M = E(\delta x) = E(\delta\hat{x}) \tag{4.5}$$

This assumption is valid to within second-order effects which may be seen by expanding the perturbation equation (4.2) through second order terms and subtracting from a second-order expansion of (3.7). An augmented matrix $P_T$ is formed as

$$P_T = \begin{bmatrix} E(\delta x\delta x^T) & E(\delta x\delta\hat{x}^T) \\ E(\delta\hat{x}\delta x^T) & E(\delta\hat{x}\delta\hat{x}^T) \end{bmatrix} = \begin{bmatrix} P & P_c \\ P_c^T & P_e \end{bmatrix} \tag{4.6}$$

At this point an important observation may be made regarding the differences in optimal linear filtering for linear versus nonlinear systems. For optimal linear filtering in linear systems, the orthogonal projection lemma [31] requires that

$$E[\delta\hat{x}(\delta x-\delta\hat{x})^T] = 0 \tag{4.7}$$

Equation (4.7) states that the estimate and the error in the estimate are uncorrelated and implies that $P_c$ and $P_e$ are equal. However, because of the inherent nonlinear nature of the problem in the shaping of the nominal trajectory, the equivalence of $P_c$ and $P_e$ is not assumed. It is shown later that $P_c$ and $P_e$ are equal only under specialized conditions. These conditions will be discussed in detail after the constraint equations on P, $P_c$, and $P_e$ have been derived.

After substitution of (4.1) into (3.7) and (4.2), one obtains

$$\delta\dot{x} = F\delta x - GK_u\delta\hat{x} + w \tag{4.8}$$

$$\delta\dot{\hat{x}} = K_eH_e\delta x + (F - GK_u - K_eH_e)\delta\hat{x} + K_e v \tag{4.9}$$

Equations (4.8) and (4.9) are in the form

$$\delta\dot{x}_T = A\delta x_T + Bn_T \tag{4.10}$$

where

$$x_T = \begin{bmatrix} \delta x \\ \delta \hat{x} \end{bmatrix} \quad n_T = \begin{bmatrix} w \\ v \end{bmatrix} \tag{4.11}$$

and

$$A = \begin{bmatrix} F & -GK_u \\ K_eH_e & F - GK_u - K_eH_e \end{bmatrix} \quad B = \begin{bmatrix} I & 0 \\ 0 & K_e \end{bmatrix} \tag{4.12}$$

Because (4.10) is linear, $P_T$ satisfies

$$\dot{P}_T = AP_T + P_TA^T + BQ_TB^T \tag{4.13}$$

The noises w and v are assumed independent so that

$$Q_T = \begin{bmatrix} Q_w & 0 \\ 0 & Q_v \end{bmatrix} \tag{4.14}$$

Substituting (4.11), (4.12), and (4.14) into (4.13), the following equations for P, $P_c$, and $P_e$ are obtained.

$$\dot{P} = FP + PF^T - GK_uP_c^T - P_cK_u^TG^T + Q_w \tag{4.15}$$

$$\dot{P}_c = FP_c + P_cF^T - GK_uP_e - P_cK_u^TG^T + PH_e^TK_e^T - P_cH_e^TK_e^T \tag{4.16}$$

$$\dot{P}_e = FP_e + P_eF^T - GK_uP_e - P_eK_u^TG^T - K_eH_eP_e - P_eH_e^TK_e^T \tag{4.17}$$

$$+ K_eH_eP_c + P_c^TH_e^TK_e^T + K_eQ_vK_e^T$$

For the Kalman filter, the filter gain is

$$K_e = \tilde{P}H_e^T Q_v^{-1} \tag{4.18}$$

where

$$\tilde{P} = E[(\delta x-\delta \hat{x})(\delta x-\delta \hat{x})^T] \tag{4.19}$$

The filter gain $K_e$ becomes after simplification

$$K_e = (P - P_c - P_c^T + P_e)H_e^T Q_v^{-1} \tag{4.20}$$

Note that $\tilde{P}$ is the error covariance of the estimate. Substituting (4.20) into (4.15), (4.16), and (4.17), the equations become

$$\dot{P} = FP + PF^T - GR_1^{-1}G^T P_u P_c^T - P_c P_u G R_1^{-1} G^T + Q_w \tag{4.21}$$

$$\dot{P}_c = FP_c + P_c F^T - GR_1^{-1}G^T P_u P_c - P_c P_u G R_1^{-1} G^T \tag{4.22}$$
$$+ (P - P_c)H_e^T Q_v^{-1} H_e (P - P_c - P_c^T + P_e)$$

$$\dot{P}_e = FP_e + P_e F^T - GR_1^{-1}G^T P_u P_e - P_e P_u G R_1^{-1} G^T \tag{4.23}$$
$$+ (P - P_c - P_c^T + P_e)H_e^T Q_v^{-1} H_e (P - P_c^T)$$
$$+ (P_c^T - P_e)H_e^T Q_v^{-1} H_e (P - P_c - P_c^T + P_e)$$

A condition under which $P_c$ and $P_e$ are equal may be determined by examination of (4.21)-(4.23). If $P_c(t_o)$ is equal to $P_e(t_o)$, then it is obvious that (4.22) and (4.23) are equal for all t. This assumption is made in all the subsequent development and (4.21)-(4.23) reduce to

$$\dot{P} = FP + PF^T - GR_1^{-1}G^TP_uP_e - P_eP_uGR_1^{-1}G^T + Q_w \tag{4.24}$$

$$\dot{P}_e = FP_e + P_eF^T - GR_1^{-1}G^TP_uP_e - P_eP_uGR_1^{-1}G^T + (P - P_e)H_e^TQ_v^{-1}H_e(P - P_e) \tag{4.25}$$

For linear systems, (4.24) and (4.25) reduce to the equations obtained for the linear combined estimation and control problem. However, if the plant is nonlinear, then P, $P_e$, and the filter gain matrix $K_e$ are influenced by the choice of nominal trajectory.

An equation for M is obtained by first expanding (3.7) by components, substituting for $\delta u$ and $K_u$, and then taking expectations. These operations yield

$$\dot{M}_j = \sum_{i=1}^{n} F_{ji}M_i - \sum_{i=1}^{m}\sum_{r=1}^{n} G_{ji}[R_1^{-1}G^TP_u]_{ir}M_r + \frac{1}{2}\sum_{k=1}^{m}\sum_{i=1}^{m}\sum_{r=1}^{n}\sum_{q=1}^{n} G_{jik}[R_1^{-1}G^TP_u]_{ir}[R_1^{-1}G^TP_u]_{kq}P_{e_{rq}} + \frac{1}{2}\sum_{k=1}^{n}\sum_{i=1}^{n} F_{jik}P_{ik} - \sum_{k=1}^{n}\sum_{i=1}^{m}\sum_{r=1}^{n} T_{jik}[R_1^{-1}G^TP_u]_{ir}P_{e_{kr}} \tag{4.26}$$

for $j = 1,2,\ldots\ldots,n$

Substituting (4.5) and (4.6) into the performance index, one obtains

$$J = \frac{1}{2}\bar{x}^T(t_f)S\bar{x}(t_f) + \bar{x}^T(t_f)SM(t_f) + \frac{1}{2}\text{ trace } SP(t_f) \tag{4.27}$$

$$+ \int_{t_o}^{t_f} \Big(\frac{1}{2}\bar{x}^TQ_1\bar{x} + \bar{x}^TQ_1M + \frac{1}{2}\text{ trace } Q_1P + \frac{1}{2}\bar{u}^TR_1\bar{u}$$

$$-\bar{u}^TG^TP_uM + \frac{1}{2}\text{ trace } G^TP_uP_eP_uR_1^{-1}G\Big)dt$$

The nonlinear combined estimation and control problem has been reduced to the minimization of (4.27) subject to the $(3n^2 + 7n)/2$ constraints given by (3.6), (4.26), (4.24), (4.25), and (3.11). Boundary conditions for the problem become

$$\bar{x}(t_o) = x_o \quad M(t_o) = 0 \quad P(t_o) = 0$$

$$P_e(t_o) = 0 \qquad \text{and} \qquad P_u(t_f) = S$$

The method used in obtaining this deterministic calculus of variations problem is approximate, but stochastic effects up to second-order have been included in obtaining the constraints. An important feature of the trajectory optimization method is that a separation principle has not been arbitrarily invoked. The structure of the resulting system is shown in Figure 13.

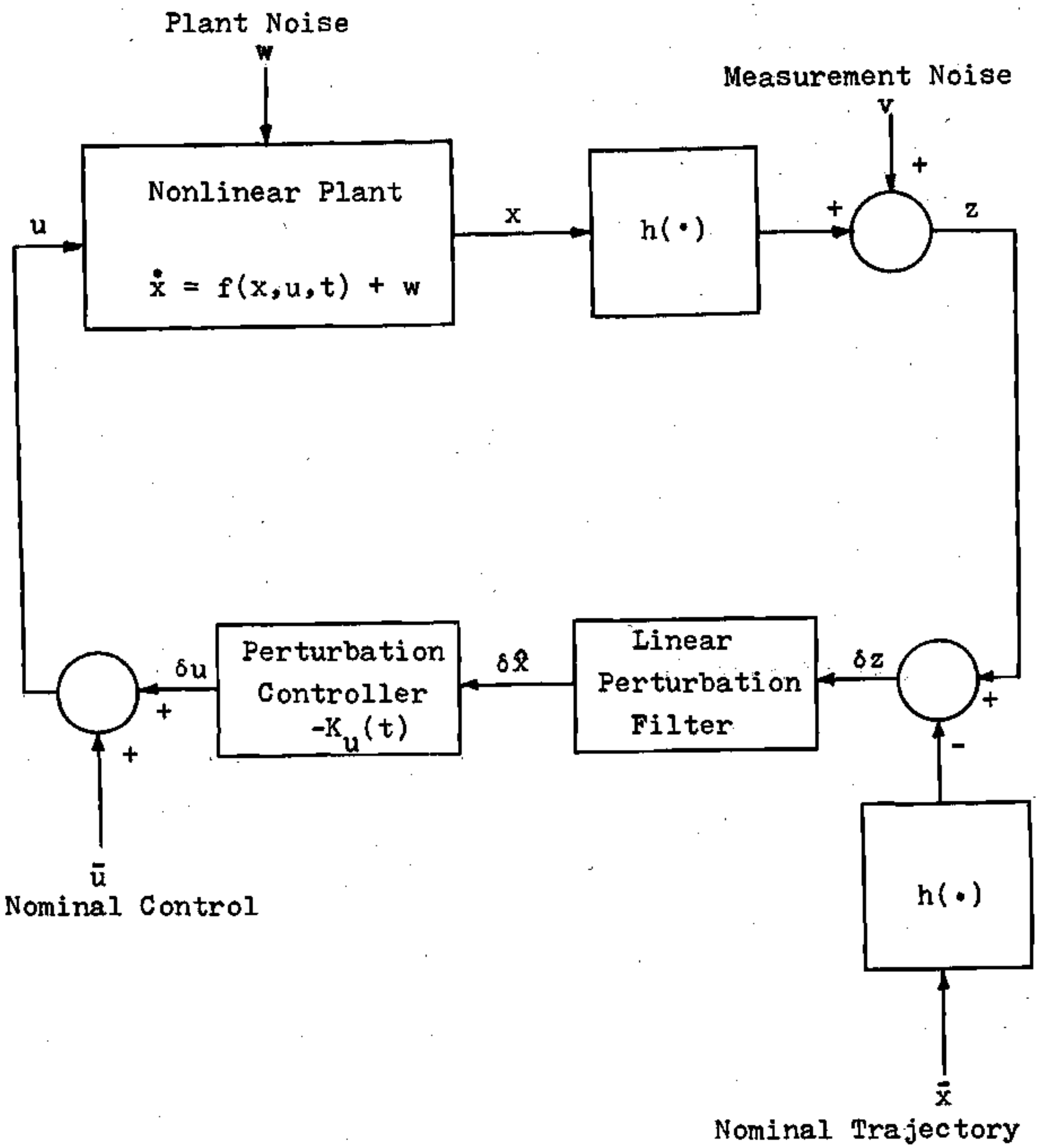


Figure 13. The Estimation and Control System Structure

## Application to the General First-order System

The theory derived in the previous section is applied to the general first-order nonlinear system. Adjoining the constraints in (3.6), (4.26), (4.24), (4.25), and (3.11) to the performance index in (4.27) by Lagrange multipliers $\bar{\lambda}$, $\lambda_M$, $\lambda_P$, $\lambda_{P_e}$, and $\lambda_{P_u}$, the Hamiltonian becomes

$$H = \tfrac{1}{2}Q_1\bar{x}^2 + Q_1\bar{x}M + \tfrac{1}{2}Q_1P + \tfrac{1}{2}R_1\bar{u}^2 - G_u\bar{u}MP_u + \frac{1}{2}\,\frac{G_u^2P_u^2P_e}{R_1} \qquad (4.28)$$

$$+ \bar{\lambda}f(\bar{x},\bar{u},t) + \lambda_M\Big(F_xM - \frac{G_u^2P_uM}{R_1} + \tfrac{1}{2}F_{xx}P + \frac{1}{2}\,\frac{G_{uu}G_u^2P_u^2P_e}{R_1^2}$$

$$\frac{-F_{xu}P_uP_e}{R_1}\Big) + \lambda_P\Big(2F_xP - \frac{2G_u^2P_uP_e}{R_1} + Q_w\Big)$$

$$+ \lambda_{P_e}\Big(2F_xP_e - \frac{2G_u^2P_uP_e}{R_1} + \frac{H_x^2}{Q_v}(P - P_e)^2\Big)$$

$$+ \lambda_{P_u}\Big(-2F_xP_u + \frac{G_u^2P_u^2}{R_1} - Q_1\Big)$$

The canonical equations become

$$\dot{\bar{x}} = \frac{\partial H}{\partial\bar{\lambda}} \qquad \dot{M} = \frac{\partial H}{\partial\lambda_M} \qquad \dot{P} = \frac{\partial H}{\partial\lambda_P} \qquad \dot{P}_e = \frac{\partial H}{\partial\lambda_{P_e}} \qquad \dot{P}_u = \frac{\partial H}{\partial\lambda_{P_u}}$$

$$-\dot{\bar{\lambda}} = \frac{\partial H}{\partial\bar{x}} \qquad -\dot{\lambda}_M = \frac{\partial H}{\partial M} \qquad -\dot{\lambda}_P = \frac{\partial H}{\partial P} \qquad -\dot{\lambda}_{P_e} = \frac{\partial H}{\partial P_e} \qquad -\dot{\lambda}_{P_u} = \frac{\partial H}{\partial P_u}$$

and $\dfrac{\partial H}{\partial\bar{u}} = 0$

Applying these canonical equations to (4.28), one obtains the following set of equations.

$$\dot{\bar{x}} = f(\bar{x},\bar{u},t) \tag{4.29}$$

$$\dot{M} = F_x M - \frac{G_u^2 P_u M}{R_1} + \frac{1}{2}F_{xx}P + \frac{1}{2}\,\frac{G_{uu}G_u^2 P_u^2 P_e}{R_1^2} - \frac{F_{xu}P_u P_e}{R_1}$$

$$\dot{P} = 2F_x P - \frac{2G_u^2 P_u P_e}{R_1} + Q_w$$

$$\dot{P}_e = 2F_x P_e - \frac{2G_u^2 P_u P_e}{R_1} + \frac{H_x^2}{Q_v}(P - P_e)^2$$

$$\dot{P}_u = -2F_x P_u + \frac{G_u^2 P_u^2}{R_1} - Q_1$$

$$-\dot{\bar{\lambda}} = Q_1\bar{x} + Q_1 M + \frac{\partial}{\partial\bar{x}}\Big[\frac{1}{2}\,\frac{G_u^2 P_u^2 P_e}{R_1} - G_u\bar{u}MP_u + \bar{\lambda}f(\bar{x},\bar{u},t)$$

$$+ \lambda_M\Big(F_x M - \frac{G_u^2 P_u M}{R_1} + \frac{1}{2}F_{xx}P + \frac{1}{2}\,\frac{G_{uu}G_u^2 P_u^2 P_e}{R_1^2} - \frac{F_{xu}P_u P_e}{R_1}\Big)$$

$$+ \lambda_P\Big(2F_x P - \frac{2G_u^2 P_u P_e}{R_1}\Big) + \lambda_{P_u}\Big(-2F_x P_u + \frac{G_u^2 P_u^2}{R_1} - Q_1\Big)$$

$$+ \lambda_{P_e}\Big(2F_x P_e - \frac{2G_u^2 P_u P_e}{R_1} + \frac{H_x^2}{Q_v}(P - P_e)^2\Big)\Big]$$

$$-\dot{\lambda}_M = Q_1\bar{x} - G_u\bar{u}P_u + \lambda_M\Big(F_x - \frac{G_u^2 P_u}{R_1}\Big)$$

$$-\dot{\lambda}_P = \frac{1}{2}Q_1 + \frac{1}{2}\lambda_M F_{xx} + 2\lambda_P F_x + \lambda_{P_e}\frac{H_x^2}{Q_v}(2P - 2P_e)$$

$$-\dot{\lambda}_{P_e} = \frac{1}{2}\,\frac{G_u^2 P_u^2}{R_1} + \lambda_M\left(\frac{1}{2}\,\frac{G_{uu}G_u^2 P_u^2}{R_1^2} - \frac{F_{xu}P_u}{R_1}\right) - 2\lambda_P\frac{G_u^2 P_u}{R_1}$$

$$+ \lambda_{P_e}\left(2F_x - \frac{2G_u^2 P_u}{R_1} + \frac{H_x^2}{Q_v}(2P_e - 2P)\right)$$

$$-\dot{\lambda}_{P_u} = -G_u\bar{u}M + \frac{G_u^2 P_u P_e}{R_1} + \lambda_M\left(\frac{-G_u^2 M}{R_1} + \frac{G_{uu}G_u^2 P_u P_e}{R_1^2} - \frac{F_{xu}P_e}{R_1}\right)$$

$$-2\lambda_P\frac{G_u^2 P_e}{R_1} - 2\lambda_{P_e}\frac{G_u^2 P_e}{R_1} + \lambda_{P_u}\left(-2F_x + \frac{2G_u^2 P_u}{R_1}\right)$$

$$R_1\bar{u} + \frac{\partial}{\partial\bar{u}}\Big[-G_u\bar{u}MP_u + \frac{1}{2}\,\frac{G_u^2 P_u^2 P_e}{R_1} + \bar{\lambda}f(\bar{x},\bar{u},t)$$

$$+ \lambda_M\left(F_x M - \frac{G_u^2 P_u M}{R_1} + \frac{1}{2}F_{xx}P + \frac{1}{2}\,\frac{G_{uu}G_u^2 P_u^2 P_e}{R_1^2} - \frac{F_{xu}P_u P_e}{R_1}\right)$$

$$+ \lambda_P\left(2F_x P - \frac{2G_u^2 P_u P_e}{R_1}\right) + \lambda_{P_e}\left(2F_x P_e - \frac{2G_u^2 P_u P_e}{R_1}\right)$$

$$+ \lambda_{P_u}\left(-2F_x P_u + \frac{G_u^2 P_u^2}{R_1}\right)\Big] = 0$$

where

$$H_x = \left.\frac{\partial h}{\partial x}\right|_{\substack{x=\bar{x}\\ u=\bar{u}}}$$

The boundary conditions are the same as in (2.24) and (3.14) with the addition of

$$P_e(t_o) = 0 \quad \lambda_{P_e}(t_f) = 0 \tag{4.30}$$

If the plant is linear, the equations (4.29) determine the deterministic optimal trajectory, along with the regular Riccati controller and Kalman filter. For nonlinear systems, the nominal trajectory, nominal control, perturbation controller, and perturbation estimator are chosen simultaneously to minimize the performance index.

## A Numerical Example

To determine the improvement in performance obtained for the combined estimation and control problem by trajectory optimization, a nonlinear first-order example was studied. The plant and performance index chosen are given in (2.25) and (2.26). In addition, the measurement is

$$z = x + v \tag{4.31}$$

where v is zero-mean white noise with power spectral density of 0.1. Trajectory optimization as determined by (4.29) was applied to the plant as given above. The resulting two-point boundary value problem was solved on the digital computer by using the gradient method. In Figure 14, a plot is given of the nominal controls corresponding to various values of $Q_w$. Figure 15 shows the nominal trajectories that correspond to the nominal controls. The perturbation feedback gains for different values of $Q_w$ are given in Figure 16.

To make comparisons between the trajectory optimized system and the systems obtained by linearization about the deterministic optimal trajectory, a Monte Carlo simulation was performed using 100 runs. The system using the deterministic optimal trajectory was implemented according to the second-variation method and also according to (3.17). The density function of wd was again selected according to (2.30). The density function of vd the discrete samples used in simulating v was

$$p_{vd}(vd) = \begin{cases} 6.5\sqrt{Q_{vd}}(vd)^{12} & \text{for } -\sqrt{Q_{vd}}<vd<\sqrt{Q_{vd}} \\ 0 & \text{otherwise} \end{cases} \qquad (4.32)$$

with

$$Q_{vd} = \frac{Q_v}{T}$$

In Figure 17, a plot is given of J versus $Q_w$ for the three different methods simulated. The performance of the system utilizing trajectory optimization was

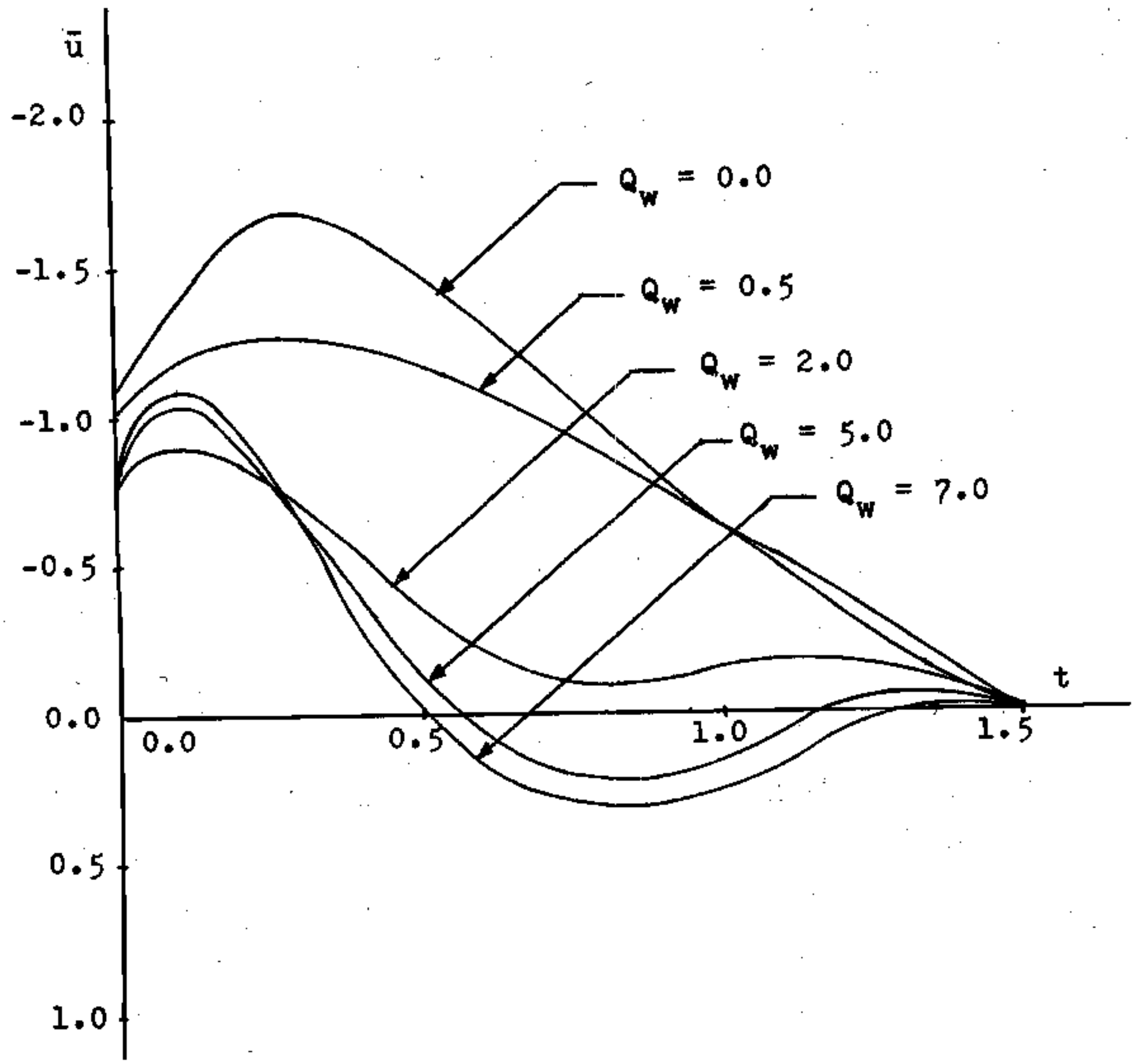


Figure 14. Stochastic Nominal Controls for the Estimation and Control Problem

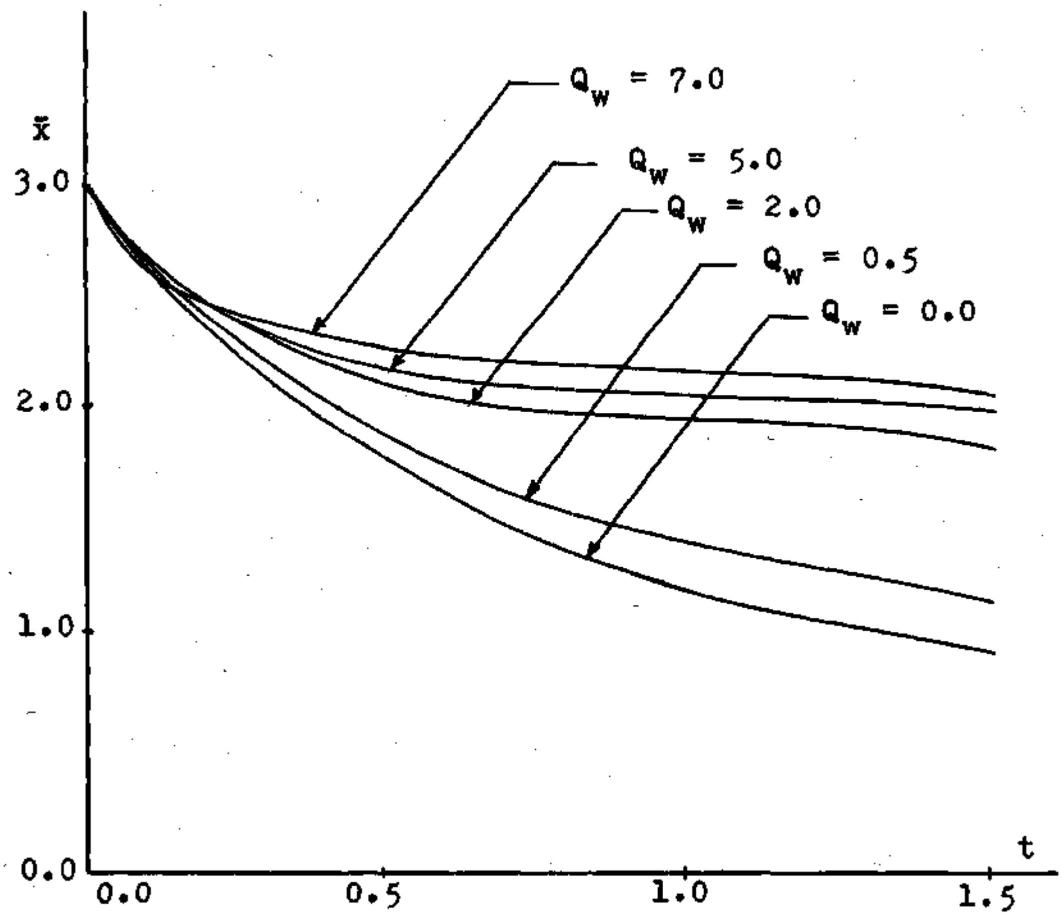


Figure 15. Stochastic Nominal Trajectories for the Estimation and Control Problem

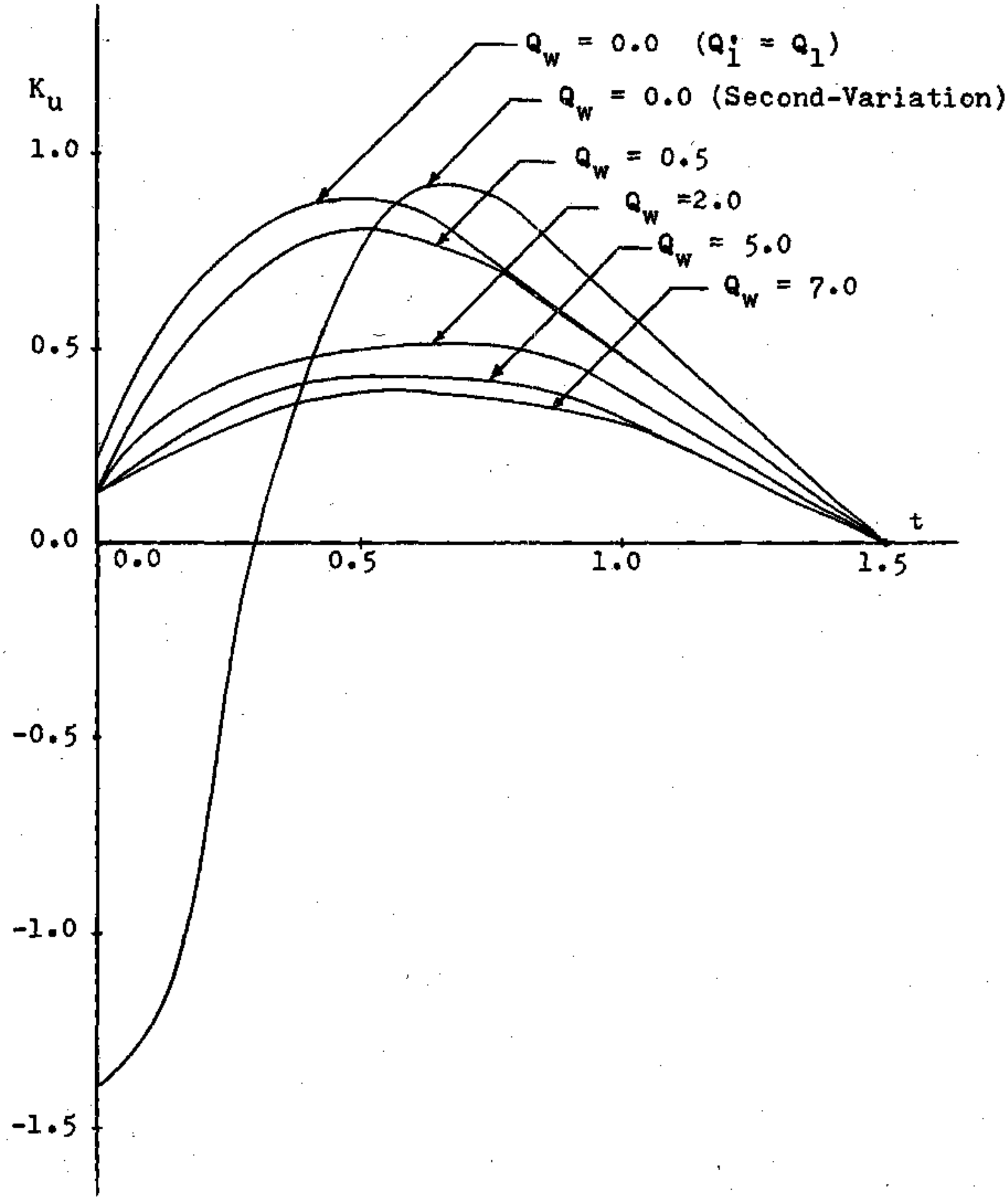


Figure 16. Stochastic Perturbation Feedback Gains for the Estimation and Control Problem

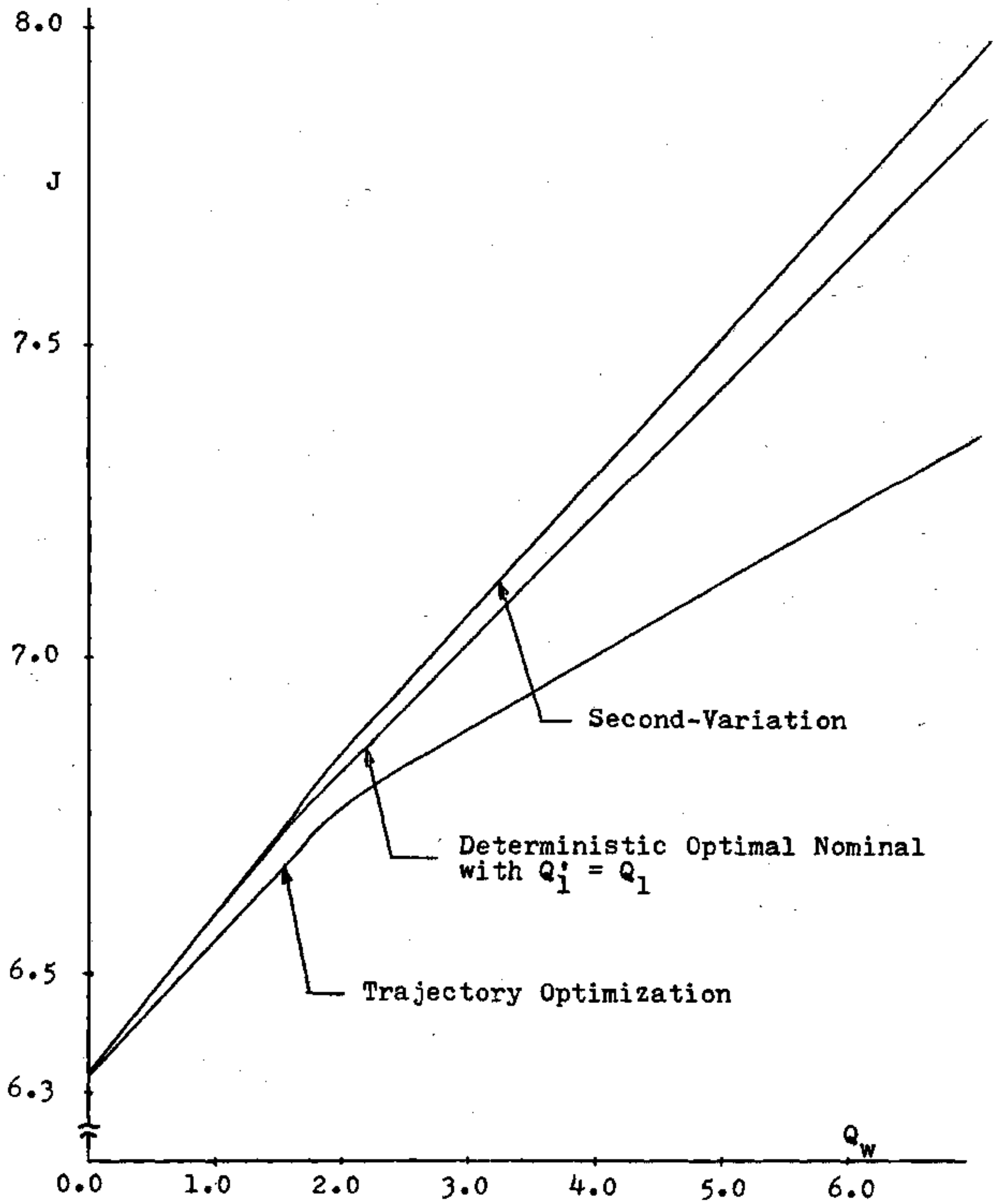


Figure 17. Performance Index Versus $Q_w$ for the Estimation and Control Problem

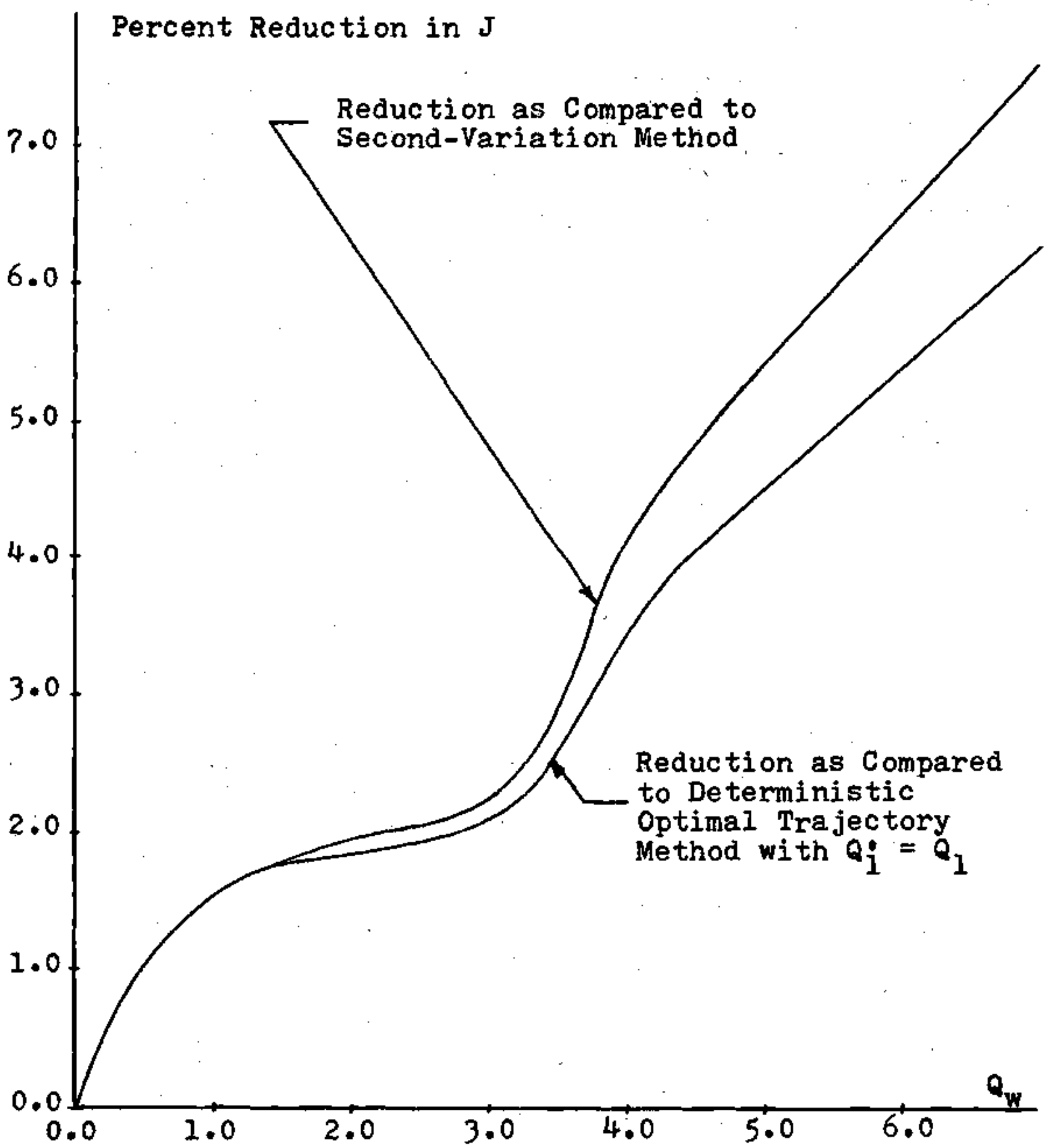


Figure 18. Percent Reduction in J Versus $Q_w$ for the Estimation and Control Problem

superior to the system performance in which either the second-variation method with deterministic optimal nominal or the perturbation method with $Q_1' = Q_1$ and deterministic optimal nominal was employed. Figure 18 indicates the percent reduction in performance index using trajectory optimization for different values of $Q_w$. Up to 6.25 percent reduction was obtained when compared with the perturbation method with $Q_1' = Q_1$ and up to 7.60 percent reduction as compared with the second-variation method.

## Conclusions

A new and significantly improved approximate solution for the nonlinear combined estimation and control problem has been presented. Linear filtering was employed in the development of the method. Superior performance was obtained using this trajectory optimization approach as compared to two other approximate methods. It is emphasized that at no point in the development was a separation principle arbitrarily invoked.

# CHAPTER V

# TRAJECTORY OPTIMIZATION FOR ESTIMATION AND CONTROL WITH NONLINEAR FILTERING

## Introduction

This chapter describes the usefulness of trajectory optimization for the combined estimation and control problem in which a nonlinear filter is utilized in the control loop. The existing nonlinear extended Kalman filter is modified to allow control inputs. It is shown that the performance of the overall system with a nonlinear estimator may be improved by finding an improved nominal trajectory. A numerical example is presented to show the improvement in performance obtained by utilizing an improved nominal trajectory for the nonlinear filter. The application of trajectory optimization to another nonlinear filter using moment calculations is also presented.

## Extension of the Extended Kalman Filter

It was noted in Chapter I that no exact nonlinear filters exist and, consequently, that approximate nonlinear filters are generally employed. It was also pointed out that for accuracy and simplicity the nonlinear extended Kalman filter is often recommended. Although more accurate filters exist, the extended Kalman is chosen for use in the control loop for this investigation because of its improved accuracy and relative simplicity.

The extended Kalman filter for systems without control inputs is considered first. The message model is selected as

$$\dot{x} = f_1(x,t) + w \tag{5.1}$$

with the measurement model

$$z = h(x) + v \tag{5.2}$$

where w and v have been defined in (1.2) and (1.4) and are assumed to be independent. The extended Kalman filter is obtained by expanding the message and measurement models in a Taylor series about the estimate $\hat{x}$ which is assumed to be known. The series is truncated after the linear terms. The expanded message and measurement models are substituted into exact equations for the conditional mean estimate and conditional mean error variance equations obtained from the Fokker-Planck equation [31,32]. The resulting approximate estimation equation is

$$\dot{\hat{x}} = f_1(\hat{x},t) + C\hat{H}Q_v^{-1}[z - h(\hat{x})] \tag{5.3}$$

where

$$\hat{H} = \left(\frac{\partial h}{\partial x}\right)^T \Bigg|_{x=\hat{x}}$$

and

$$C = E[(x-\hat{x})(x-\hat{x})^T] \tag{5.4}$$

The error variance defined by (5.4) satisfies the stochastic differential equation

$$\dot{C} = \hat{F}_1 C + C\hat{F}_1^T + Q_w - C\hat{H}Q_v^{-1}\hat{H}^T C \tag{5.5}$$

where

$$\hat{F}_1 = \left(\frac{\partial f_1}{\partial x}\right)^T \Bigg|_{x=\hat{x}}$$

It is pointed out that the variance C may not be precalculated because it depends upon the estimate $\hat{x}$. Therefore, both (5.3) and (5.5) are used on-line for estimation of the state x.

So that the extended Kalman filter may be used in a control situation, the basic filter is modified to permit the inclusion of a control u in the message model. The message model is assumed to be

$$\dot{x} = f(x,u,t) + w \tag{5.6}$$

with the measurement equation remaining as in (5.2). It is further assumed that

$$\frac{\partial f_j(x,u,t)}{\partial x_i \partial u_k} = 0 \quad \text{for} \quad \begin{array}{l} j = 1,2,\ldots\ldots,n \\ i = 1,2,\ldots\ldots,n \\ k = 1,2,\ldots\ldots,m \end{array} \tag{5.7}$$

The restriction in (5.7) means that none of the elements of the control may be state-dependent. This assumption simplifies the derivation of the variance equation for the estimation error. The estimator is constructed so that it contains the control input to the system. The equation for the estimate becomes

$$\dot{\hat{x}} = f(\hat{x},u,t) + C\hat{H}Q_v^{-1}[z - h(\hat{x})] \tag{5.8}$$

where C is again the variance matrix of the estimation error e, where

$$e = x - \hat{x} \tag{5.9}$$

The equation which C must satisfy has to be determined. If (5.7) holds, then the message and estimation equations may be written as

$$\dot{x} = f_1(x,t) + g(u,t) + w \tag{5.10}$$

$$\dot{\hat{x}} = f_1(\hat{x},t) + g(u,t) + C\hat{H}Q_v^{-1}[z - h(\hat{x})] \qquad (5.11)$$

Subtracting (5.11) from (5.10), the error equation becomes

$$\dot{e} = f_1(x,t) - f_1(\hat{x},t) + w - C\hat{H}Q_v^{-1}[z - h(\hat{x})] \qquad (5.12)$$

Note that the control u does not affect the error equation, and that (5.12) is precisely the equation obtained when (5.3) is subtracted from (5.1). It is obvious that, for the restriction on the message model in (5.7), the variance equation for the message model with control input (5.6) is the same as for the message model in (5.1) with no control input. The requirement in (5.7) is not too restrictive because a large number of practical systems have controls which are not state-dependent.

The estimation equation in (5.8) is modified to allow perturbations from the nominal trajectory to be estimated. It is assumed that the state and control may be written as in (2.1) and (3.1). The estimate $\hat{x}$ of the state x may be written as

$$\hat{x} = \bar{x} + \delta\hat{x} \qquad (5.13)$$

The perturbation feedback controller in (3.3) is utilized and the total control u becomes

$$u = \bar{u} - K_u \delta \hat{x} \qquad (5.14)$$

Substituting (5.13) and (5.14) into (5.8), one obtains

$$\dot{\hat{x}} = \dot{\bar{x}} + \delta\dot{\hat{x}} = f(\bar{x}+\delta\hat{x}, \bar{u}-K_u\delta\hat{x}, t) + C\hat{H}Q_v^{-1}[z - h(\bar{x}+\delta\hat{x})] \qquad (5.15)$$

Subtracting (3.6) from (5.15), the perturbation estimator becomes

$$\delta\dot{\hat{x}} = f(\bar{x}+\delta\hat{x}, \bar{u}-K_u\delta\hat{x}, t) - f(\bar{x}, \bar{u}, t) + C\hat{H}Q_v^{-1}[z - h(\bar{x}+\delta\hat{x})] \quad (5.16)$$

where the variance C of the estimation error satisfies (5.5).

Trajectory Optimization for the Extended Kalman Filter

If the method of trajectory optimization were applied to the nonlinear system with (5.16) as the estimator, the nonlinear nature of the estimation equation would require, in general, the knowledge of differential equations for an infinite number of moments of $\delta\hat{x}$. This fact makes the strict application of trajectory optimization as given in the three previous chapters prohibitive. However, an examination of (5.16) reveals that a Taylor series expansion of the equation about $\bar{x}$ and $\bar{u}$ produces the linear estimator of Chapter IV when nonlinear terms are neglected. Thus, it would seem plausible that the nominal trajectory obtained by trajectory optimization for the linear estimator might be an improved nominal trajectory for the overall system using the extended Kalman filter. The following estimation and control algorithm is thus proposed for use with the extended Kalman filter:

1. Using the optimized nominal trajectory and nominal control for the linear filter case of Chapter IV, calculate the Riccati perturbation feedback gains off-line using (3.4) and (3.11).
2. Employ (5.16) and (5.5) on-line to estimate perturbations about the nominal.
3. Use (5.14) to construct the total control for the system.

## A Numerical Example

To determine what effects the introduction of non-linear filtering into the control loop produced, the non-linear example previously described in Chapters II, III, and IV was considered. The plant was described by (2.26), the performance index by (2.25), and the measurement equation by (4.31). The nominal controls, nominal trajectories, and perturbation feedback gains as given in Figures 14, 15, and 16, respectively, were used in the application of (5.16) and (5.5) on-line. A Monte Carlo simulation with 100 runs was performed to calculate the performance indices associated with the system for several values of $Q_w$. The measurement noise was held constant with a power spectral density of 0.1.

In Figure 19, a plot is shown of J versus $Q_w$. The upper dotted curve indicates the performance obtained when the deterministic optimal trajectory was employed with the extended Kalman filter. It may be seen that introducing the nonlinear extended Kalman filter to estimate perturbations about the deterministic optimal trajectory produced some improvement in the system performance. The lower dotted curve is a plot of J obtained when the extended Kalman filter was used with the improved nominal trajectories of Chapter IV. Obviously, the increase in performance was much more significant with an improved nominal trajectory. The lower curve is the performance reported in Chapter IV for the linear filter with the nominal trajectory specifically optimized for the overall system. In this example the linear filter with the trajectory optimized nominal performed better than the nonlinear filter using an improved nominal trajectory. This occurred because the nominal trajectory was not specifically optimized for the extended Kalman filter but was only an improved nominal trajectory.

## Nonlinear Filtering Using Moment Calculations

This section describes the application of trajectory optimization for other combined estimation and control problems which utilize nonlinear filters obtained from moment calculations. Clark [50] considered the nonlinear combined estimation and control problem for the example in the

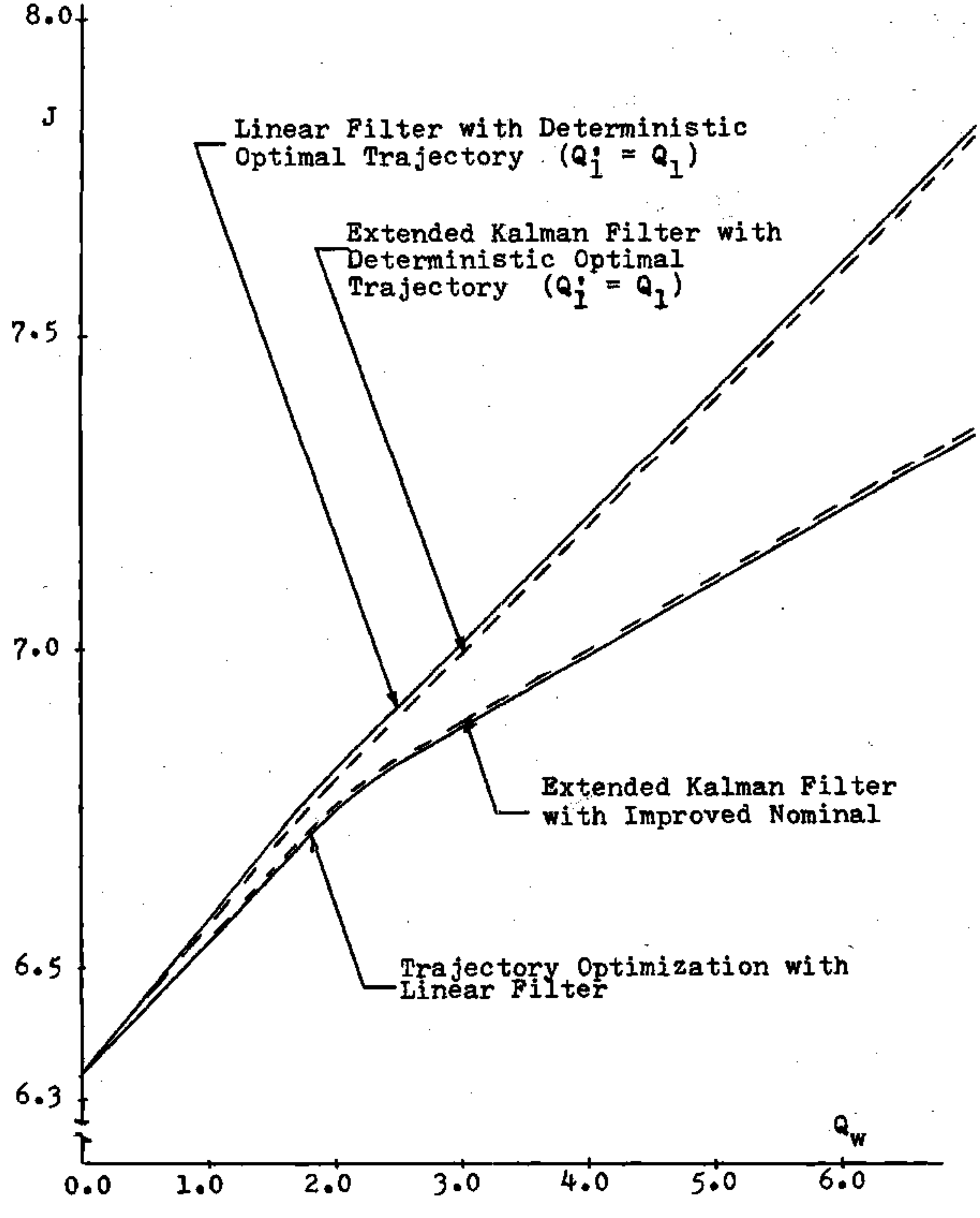


Figure 19. Performance Index Comparison for Linear and Nonlinear Filtering

previous section. Clark utilized a very accurate filter that was based on the method of moments [29,30]. The filter estimated perturbations about a nominal trajectory. Two nominal trajectories were compared with regard to the performance of the overall system obtained when the trajectories were employed for estimation and control. The first trajectory was the deterministic optimal trajectory and the second was the optimized trajectory obtained from the work described in Chapter IV of this dissertation. The performance index for the optimized trajectory was 8.9 percent less than the performance index obtained for the case in which the deterministic optimal trajectory was used. Apparently, the system performance was significantly improved when the improved nominal trajectory described in Chapter IV was utilized.

## Conclusions

The use of nonlinear filters for the combined estimation and control problem has been considered in this chapter. The conclusions drawn and the insight gained from the work described in this chapter may be summarized as follows:

1. Trajectory optimization for the linear filter provided superior performance compared to the performance obtained with the nonlinear extended Kalman filter utilizing the deterministic optimal trajectory.

2. Applying the linear filter optimized nominal to the extended Kalman filter improved the overall system performance.
3. The linear filter with the optimized nominal performed better than the nonlinear extended Kalman filter with the same optimized nominal. This was because the nominal was not specifically optimized for the extended Kalman filter.
4. Using a moment-derived nonlinear filter with the trajectory optimized for the Kalman filter improved overall system performance even though the trajectory was not specifically optimized for the moment-derived filter.
5. The performance of the nonlinear system was much more sensitive to the choice of nominal trajectory than to the accuracy of the estimator used in the control loop.

# CHAPTER VI

# CONCLUSIONS AND RECOMMENDATIONS

## Results and Conclusions

The major result of this thesis research has been the development of an improved approximate solution for the nonlinear combined estimation and control problem. The method of solution has been to approximately optimize the nominal trajectory for the nonlinear system. The only requirement on the nonlinear system is that the nonlinearity be expandable in a Taylor series and the noise enter linearly into the plant equations. In the development of trajectory optimization, the open-loop system, the closed-loop system with zero measurement noise, and the closed-loop system with both plant and measurement noise have been treated. It was shown conclusively through specific nonlinear examples that a significant improvement in system performance could be obtained by utilizing trajectory optimization.

The basic idea in developing the trajectory optimization method was to transform the stochastic optimization problem to a deterministic optimization problem in the calculus of variations. When trajectory optimization was applied to a nonlinear system, a set of nonlinear differential equations of higher order with two-point boundary conditions had to be solved. In effect, increased accuracy was obtained at the expense of higher dimensionality. However, in view of current computer capabilities, systems up to fourth or fifth order could be handled by trajectory optimization. An important feature of trajectory optimization was that the nominal trajectory and nominal control may be calculated off-line and stored in the digital computer for on-line operation. Therefore, the two-point boundary value problem did not have to be solved on-line.

In the final stage of the research, the nonlinear combined estimation and control problem was considered. Both linear and nonlinear filters were utilized. As mentioned above, trajectory optimization provided a significant increase in the performance of the overall control system. A very important conclusion was that the system performance appeared to be much more sensitive to the choice of nominal trajectory than to the selection of a nonlinear filter of greater accuracy.

## Recommendations for Further Work

There are several possible directions in which one might proceed to extend the ideas presented in this thesis. Although increased accuracy was realized by the method of trajectory optimization, further improvement might be obtained by expanding the system perturbation equations to higher-order such that more moments of $\delta x$ would be included.

Expansion to higher-order would require more equations which would limit the practicality of using higher-order expansion. Rowland and Holmes [51] have used the idea of taking multiple variations in the plant state to obtain increased accuracy in numerical integration algorithms. It is possible that this variational approach might be applicable to the trajectory optimization problem.

Another possible application of trajectory optimization would be in the nonlinear filtering problem where no control is present for the message model. Often a nominal trajectory is utilized in nonlinear filtering. The nominal is usually taken as the trajectory resulting when the system state is allowed to evolve from its initial state in the absence of the system noise [32]. It would seem that trajectory optimization might be applied to this problem to obtain an improved nominal trajectory about which perturbations could be estimated. The improvement would be in the form of reduced estimation error.

This thesis has presented a new trajectory optimization technique for solving the combined estimation and control problem. It has been demonstrated through numerical examples that significant improvement is obtained when the method of trajectory optimization is used rather than simply linearizing about the deterministic optimal trajectory. Possible extensions of the work performed in this thesis research are indicated in this section.

# BIBLIOGRAPHY

1. N. Wiener, Extrapolation, Interpolation, and Smoothing of Stationary Time Series with Engineering Applications, Wiley, New York, 1949.

2. A. N. Kolmogorov, "Interpolation und von Stationaren Zufalligen Folgen," Bulletin of the Academy of Science (U.S.S.R.), Ser. Math. Vol. 5, pp. 3-14, 1941.

3. R. E. Kalman and R. S. Bucy, "New Results in Linear Filtering and Prediction Theory," Transactions of the ASME, Journal of Basic Engineering, Series D, Vol. 83, pp. 95-108, 1961.

4. R. L. Stratonovich, "Conditional Markov Processes," Theory of Probability and Applications, 1960.

5. H. J. Kushner, "On the Differential Equations of Conditional Probability Density Functions with Applications to Optimal Stochastic Control Theory," Journal of Mathematical Analysis and Applications, Vol. 8, pp. 332-344.

6. H. J. Kushner, "On the Differential Equations Satisfied by Conditional Probability Densities of Markov Processes," SIAM Journal of Control, Vol. 2, pp. 106-119, 1964.

7. R. S. Bucy, "Nonlinear Filtering Theory," IEEE Transactions on Automatic Control, Vol. AC-10, p. 198, 1965.

8. W. M. Wonham, "Stochastic Problems in Optimal Control," IEEE Convention Record, Part 2, 11, pp. 114-124, 1963.

9. R. L. Kashyap, "On the Partial Differential Equation for the Conditional Probability Distribution for Nonlinear Dynamic Systems with Noisy Observations," Tech. Report 432, Cruft Laboratory, Harvard University, Cambridge, Massachusetts, 1963.

10. J. R. Fisher, "Optimal Nonlinear Filtering," Advances in Control Systems, Vol. 5, 1967.

11. Y. C. Ho and R. C. K. Lee, "A Bayesian Approach to Problems in Stochastic Estimation and Control," IEEE Transactions on Automatic Control, Vol. AC-9, pp. 333-339, 1964.

12. A. H. Jazwinski, "Nonlinear Filtering with Discrete Observations," Paper No. 66-38, AIAA 3rd Aerospace Sciences Meeting, New York, 1966.

13. R. Bellman, "On the Foundations of a Theory of Stochastic Variational Problems," Proceedings, Symposium of Applied Mathematics, Vol. 13, American Mathematical Society, pp. 275-286, 1962.

14. L. S. Pontryagin, et al., The Mathematical Theory of Optimal Processes, Interscience Publishers, New York, New York, 1962

15. E. F. Mischenko and L. S. Pontryagin, "A Statistical Theory of Control," Dokl. Akad. Nauk. SSSR, 128, pp. 890-892, 1959.

16. A. N. Kolmogorov, E. F. Mischenko, and L. S. Pontryagin, "On a Probability Problem of Optimal Control," Dokl. Akad. Nauk. SSSR, 145 (5) pp. 993-995, 1962.

17. J. J. Florentin, "Optimal Control of Continuous-Time Markov Stochastic Systems," Journal of Electronics and Control, Vol. 10, pp. 473-488, 1961.

18. W. M. Wonham, "Stochastic Problems in Optimal Control," IEEE International Convention Record, Part 2, pp. 114-124, March 1963.

19. L. Meier III, "Combined Optimal Control and Estimation Theory," Contractor Report NASA CR-426, Contract 2-2457, SRI Project 5237, Stanford Research Institute, Menlo Park, California, April 1966.

20. A. A. Feldbaum, "Dual Control Theory I," Avtomatika i Telemehkanika, September 1960.

21. P. D. Joseph and J. T. Tou, "On Linear Control Theory," AIEE Transactions on Applications and Industry, Vol. 80, pp. 193-196, September 1961.

22. D. L. Alspach and H. W. Sorenson, "A Separation Principle for Nongaussian Noise," International Journal of Control, (To Appear).

23. R. E. Curry, "A Separation Theorem for Nonlinear Measurements," IEEE Transactions on Automatic Control, Vol. AC-14, No. 5, pp. 561-564, October 1969.

24. C. Striebel, "Sufficient Statistics in the Optimum Control of Stochastic Systems," Journal of Math. Anal. and Appl., 12, No. 3. pp. 576-592, 1965.

25. W. M. Wonham, "On the Separation Theorem of Stochastic Control," SIAM Journal on Control, 6, No. 3, pp. 312-326, 1968.

26. R. Gran, "A Separation Theorem for Linear Continuous Time Systems with Nonlinear Controls and Free Terminal Time Performance Indices," SIAM Journal on Control, (To Appear).

27. R. Gran, "An Approach to the Separation of Control and Estimation for Nonlinear Stochastic Systems," Proceedings of the Symposium on Nonlinear Estimation Theory and Its Applications, pp. 215-219, San Diego, Calif., September 1970.

28. H. W. Sorenson and A. R. Stubberud, "Nonlinear Filtering by Approximation of the A Posteriori Density," International Journal of Control 8, pp. 33-51, 1968.

29. Te-Son Kuo, "A Moment Technique for Suboptimal Adaptive Nonlinear Filtering," Ph.D. Thesis, Georgia Institute of Technology, December 1969.

30. Te-Son Kuo and J. R. Rowland, "A Moment Technique for Suboptimal Adaptive Nonlinear Filtering," 1970 Southwestern IEEE Conference (SWIEEECO) Record, pp. 105-109, April 1970.

31. Andrew H. Jazwinski, Stochastic Processes and Filtering Theory, Academic Press, 1970.

32. Andrew P. Sage and James L. Melsa, Estimation Theory with Applications to Communications and Control, McGraw-Hill, 1971.

33. A. H. Jazwinski, Filtering for Nonlinear Dynamical Systems," IEEE Transactions on Automatic Control, Vol. AC-11, pp. 765-766, 1966.

34. A. H. Jazwinski, "Stochastic Processes with Application to Filtering Theory," Analytical Mechanics Associates, Inc., Seabrook, Maryland, Rept. No. 66-6, 1966.

35. R. W. Bass, V. D. Norum, and L. Schwartz, "Optimal Multichannel Nonlinear Filtering," Journal of Math. Anal. and Appl., 16, pp. 152-164, 1966.

36. L. Schwartz, "Approximate Continuous Nonlinear Minimal-Variance Filtering," Hughes Aircraft Company, Space Systems Division, El Segundo, California, SSD 60472R, Report Number 18, December 1966.

37. Y. Sunahara, "An Approximation Method of State Estimation for Nonlinear Dynamical Systems," Brown University, Center for Dynamical Systems, Providence, Rhode Island, Tech. Report 67-8, December 1967.

38. L. Schwartz and E. B. Stear, "A Computational Comparison of Several Nonlinear Filters," IEEE Transactions on Automatic Control, Vol. AC-13, pp. 83-86, February 1968.

39. A. E. Bryson and Y. C. Ho, Applied Optimal Control, Ginn-Blaisdell, Waltham, Massachusetts, 1969.

40. H. J. Kushner, "Near Optimal Control in the Presence of Small Stochastic Perturbations," Proceedings of the 1964 Joint Automatic Control Conference, Stanford University, pp. 392-396, June 1964.

41. W. J. Culver and M. D. Mesarovic, "Dynamic Statistical Linearization," AEEE Transactions on Computing Devices, pp. 317-327, July 1963.

42. Y. Sunahara and A. Ohsumi, "Stochastic Estimation and Optimal Control for Nonlinear Dynamical Systems with State-Dependent Noise," Proceedings of the Symposium on Nonlinear Estimation Theory and Its Applications, pp. 230-333, September 1970.

43. W. H. Denham, "Choosing the Nominal Path for a Dynamic System with Random Forcing Functions to Optimize Statistical Performance," Tr 449, Division of Eng. and App. Physics, Harvard University, 1964.

44. D. R. Vander Stoep, "Trajectory Shaping for the Minimization of State Variable Estimation Errors," IEEE Transactions on Automatic Control, pp. 284-286, June 1968.

45. L. Meier, J. Peschon, B. Ho, R. Larson, and R. Dressler, "Design of Guidance and Control Systems for Optimum Utilization of Information," SRI Project, NASA CR-997, Stanford Research Institute, Menlo Park, California, 1967.

46. R. Fitzgerald, "A Gradient Method for Optimizing Stochastic Systems," Ph.D Thesis, M.I.T., May 1964.

47. A. P. Sage, Optimum Systems Control, Prentice Hall, 1968.

48. J. H. Laning, Jr. and R. H. Battin, Random Processes in Automatic Control, McGraw-Hill, 1956.

49. J. V. Breakwell, J. L. Speyer, and A. E. Bryson, "Optimization and Control of Nonlinear Systems using the Second Variation," SIAM Journal on Control, Ser. A, Vol. 1, pp. 193-223, 1963.

50. George M. Clark, Jr., "An Improved Algorithm for the Combined Estimation and Control of Nongaussian Stochastic Systems," Ph.D. Thesis, Georgia Institute of Technology, May 1971.

51. James R. Rowland and Willard M. Holmes, "A Variational Approach to Digital Integration," IEEE Transactions on Computers, August 1971.

# VITA

Robert J. Brown, Jr. was born in Reidsville, Georgia on March 25, 1945. He entered the Georgia Institute of Technology in 1963 and received the B.E.E. and M.S.E.E. degrees in 1968 and 1970, respectively. In 1968, he received a National Science Foundation Trainee Fellowship. In 1969, he worked as a research assistant in the School of Electrical Engineering under a grant from the Lockheed-Georgia Company. In 1970, he was a teaching assistant and was also the recipient of a Schlumberger Foundation Fellowship.

In August, 1967, Mr. Brown married the former Martha Gwen Randolph of Tupelo, Mississippi. He will complete the requirements for his Ph.D. in Electrical Engineering in the field of stochastic optimal control theory in May, 1971.